LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

AI 501 Mathematics for Artificial Intelligence
Quiz 07 Solutions

Name: ____________________

Campus ID: ____________________

Total Marks: 10
Time Duration: 15 minutes

Question 1 (4 marks)

From the given options, select whichever is correct. There may be more than one correct option.

1. Which of the following conditions is a condition for the convergence of gradient descent to global minimum?
   (a) The objective function must be convex.
   (b) The learning rate must be large enough to overcome noise.
   (c) The learning rate must be small enough so that the algorithm does not deviate from the global minimum.
   (d) The gradient of the function must be constant.
2. Which of the following functions is convex?
   (a) $f(x) = x^3$
   (b) $f(x) = \log(x)$ for $x > 0$
   (c) $f(x) = x^2$
   (d) $f(x) = -x^2$
3. Which of the following is a characteristic of a convex optimization problem?
   (a) The objective function is concave.
   (b) The feasible set is convex.
   (c) The optimization problem has multiple local minima.
   (d) Any local minimum is a global minimum.
4. Consider the vector-valued function $\mathbf{F}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, where $\mathbf{F}$ is differentiable. Its first-order Taylor expansion around a point $\mathbf{x}_0$ is:

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{x}_0) + J(\mathbf{F})(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

   Which of the following statements is true regarding the **Jacobian matrix** $J(\mathbf{F})(\mathbf{x}_0)$ in this context?
   (a) It represents the curvature of $\mathbf{F}$ at $\mathbf{x}_0$.
   (b) It gives the best linear approximation of $\mathbf{F}$ at $\mathbf{x}_0$.
   (c) It is used to determine whether $\mathbf{F}$ is convex at $\mathbf{x}_0$.
   (d) It provides a quadratic approximation of $\mathbf{F}$ at $\mathbf{x}_0$.

Solution:

1. (a) and (c). If the objective function is not convex, the gradient descent may converge to a local minimum. If the step size is too large, the algorithm may oscillate between two points and may overshoot the global minimum.
2. (c). None of the other functions are convex.
3. (a), (b) and (c) are not correct. By definition, a convex optimization problem must have a convex objective function with convex feasibility constraints. The fact that the problem is convex implies that any local minimum is a global minimum.

Question 2 (6 marks)

(a) Verify the below optimization problem is convex by checking the convexity of the objective function and the constraints.

(b) Solve the optimization problem.

Minimize the following quadratic function:

$$f(x, y) = (x - 1)^2 + (y - 2)^2$$

subject to the constraints:

$$x + y \leq 3$$

$$x \geq 0$$

$$y \geq 0$$

Hint: Check the minimum possible value of the objective function given it is a sum of two squares.

Solution:

(a)

$$f(x, y) = x^2 + y^2 - 2x - 4y + 5$$

Hessian:

$$H(f) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The Hessian is positive definite (determinant is positive), so $f(x, y)$ is convex.

- $x + y \leq 4$: Convex (half-plane).
- $x \geq 0$: Convex (half-plane).
- $y \geq 0$: Convex (half-plane).

(b) We minimize the objective function $f(x, y)$ now. We can rewrite the function as:

$$f(x, y) = x^2 + y^2 - 2x - 4y + 5 = x^2 + y^2 - 2x - 4y + 1 + 4$$
$$f(x, y) = x^2 - 2x + 1 + y^2 - 4y + 4 = (x - 1)^2 + (y - 2)^2$$

Since the function is strictly positive (sum of two squares), the smallest possible value it can take is 0. In fact, the function is only 0 at $x = 1$ and $y = 2$. As these values satisfy our feasibility constraints, this must be the optimal solution. Hence,

$$x^* = 1$$
$$y^* = 2$$

And:

$$f(x^*, y^*) = 0$$