EE 212 – Mathematical Foundations for Machine Learning and Data Science

Mid-Examination (Summer 2020)

INSTRUCTIONS:

- We require you to solve the exam in a single time-slot of two hours without any external or electronic assistance.
- We encourage you to solve the exam on A4 paper, use new sheet for each question and write sheet number on every sheet.
- Clearly outline all your steps in order to obtain any partial credit.
- The exam is closed book and notes. You are allowed to have one A4 sheet with you with hand-written notes on both sides. Calculators can be used.
- For the sake of completeness, we require you to write the following statement on your first page of submission: *I commit myself to uphold the highest standards of (academic) integrity.*
- If you are ready, please proceed to the next page.

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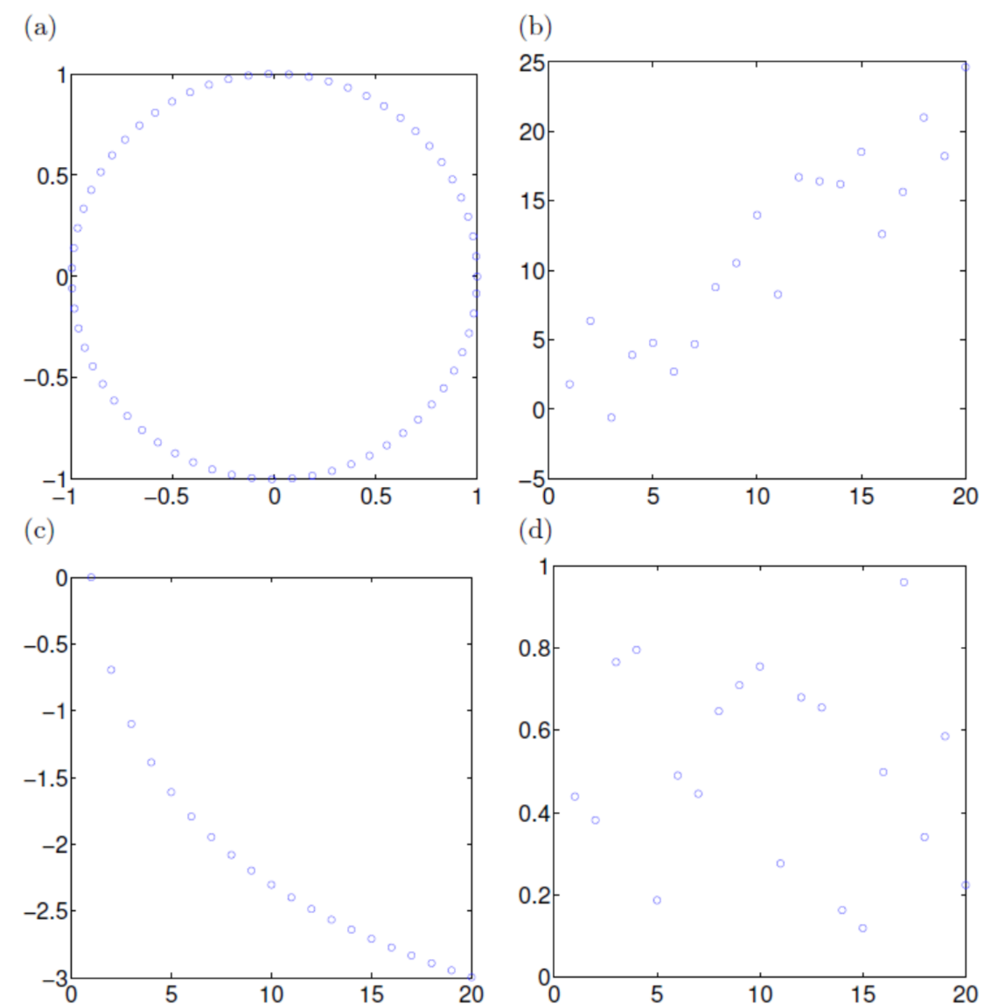
Total marks: 60 points **Duration:** 2 hours

Problem 1. (4 pts)

In each of the plot given below, we have plotted two vectors x, $y \in \mathbf{R}^n$ such that each point represents (x_i, y_i) for $i = 1, 2, \ldots, n$. In each case, determine whether the correlation coefficient ρ of two vectors is positive ($\rho > 0.5$), negative ($\rho < -0.5$) or near zero (say $|\rho| < 0.3$).



Problem 2. (6 pts)

What are the properties of a norm function? Determine 1-norm, Euclidean norm and ∞-norm of a vector $x = (7, -1, 2, -1, 3)$.

Problem 3. (10 pts)

Let $A \in \mathbf{R}^{m \times n}$ represent a contingency matrix of m cabinet members who hold n residencies (or citizenships)

$$A_{ij} = \begin{cases} 1 & \text{member } i \text{ holds residency } j \\ 0 & \text{member } i \text{ does not hold residency } j, \end{cases}$$

where we note that a member can hold any number of residencies.

(a) Give a simple formula for the n-vector g, where g_i is the total number of members holding residency j.

Give simple English description of the following expressions or quantities.

(b) 2-nd column of A.

(c) $(AA^T)_{ij}$

(d) $(A^T A)_{ij}$

Problem 4. (10 pts)

Which of following matrices are left-invertible? Provide justification to support your answer.

(a) $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \\ 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & -6 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} -9 & 0 & 7 \\ 4 & 0 & -5 \\ -1 & 0 & 6 \end{bmatrix}$

(d) $A = \begin{bmatrix} B \\ C \end{bmatrix}$, where $B \in \mathbf{R}^{n \times n}$ is a diagonal matrix with non-zero diagonal entries and $C \in \mathbf{R}^{m \times n}$ can be any matrix.

(e) $A = ab^T$, where $a, b \in \mathbf{R}^n$ for $n > 1$

Problem 5. (10 pts)

Use Gram-Schmidt process to determine the QR factorization of the matrix A given by

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 \\ 0 & 1 \end{bmatrix}.$$

Problem 6. (**6 pts**)

Consider a matrix A given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Give simple description of the relationship between x and Ax.

(b) Determine A^5.

Problem 7. (**4 pts**)
(**Independent parts**)

(a) Consider two linearly independent vectors $a_1, a_2 \in \mathbf{R}^3$. Consider a set W that contains all points of the form $\beta_1 a_1 + \beta_2 a_2 + 3$. Is W a subspace? If yes, what is the dimension of the subspace?

(b) Consider a vector space $V = \mathbf{R}^m$ and the span of the vectors $a_1, a_2, \ldots, a_n$ of the vector space V. What is the maximum possible dimension of the span of the vectors $a_1, a_2, \ldots, a_n$ of the vector space?

Problem 8. (10 pts)

Let $A \in \mathbf{R}^{3\times 2}$ be a matrix given by

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{bmatrix},$$

with singular-value decomposition given as $A = U\Sigma V^T$. The eigenvalue decomposition of $B = A^T A$ has 1 and 17 as its two non-zero eigenvalues and one of the two normalised eigenvectors is

$$q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

(a) Determine the matrix V.
(Hint: You only need to use the given information above and exploit the fact that V is an orthogonal matrix.)

(b) Determine the singular values of A, that is, the matrix Σ.
(Hint: Relate the EVD of B with the SVD of A)

(c) Determine the determinant of the matrix B.

— End of Exam —