Department of Electrical Engineering
School of Science and Engineering

EE212 Mathematical Foundations for Machine Learning and Data Science

ASSIGNMENT 1

Due Date: 23:55, Sunday, September 26, 2021. (Submit online on LMS)
Format: 12 problems, for a total of 100 marks
Instructions:

- You are allowed to collaborate with your peers but copying your colleague's solution is strictly prohibited. This is not a a group assignment. Each student must submit his/her own assignment, scanned in **a single PDF document.**
- Solve the assignment on blank A4 sheets and either scan the document using a scanner or use CamScanner proficiently.
- Upload the solved assignment on LMS in the "Assignments" tab under Assignment 01.
- Naming convention should be as follows: "Name_RollNumber_Assignment_1.pdf"
- Feel free to contact the instructor or the teaching assistants if you have any concerns.

- You represent the most competent individuals in the country, do not let plagiarism come in between your learning. In case any instance of plagiarism is detected, the disciplinary case will be dealt with according to the university's rules and regulations.

Problem 1 (8 marks)

Given the vectors below,

$$a = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Calculate the following.

(a) **[2 marks]** $2a + b$

(b) **[2 marks]** $7a - 4c$

(c) **[2 marks]** $(a \times b) \times c$, where '$\times$' denotes the cross product.

(d) **[2 marks]** $7a \cdot c$, where '$\cdot$' denotes the dot product.

Problem 2 (14 marks)

For the vectors

$$a = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(a) **[2 marks]** Compute $a^T b$.

(b) **[2 marks]** Compute $b^T a$.

(c) **[2 marks]** Compute ab^T.

(d) **[2 marks]** Compute ba^T.

(e) **[3 marks]** Describe the relationship between $a^T b$ and $b^T a$. Discuss its interpretation.

(f) **[3 marks]** Describe the relationship between ab^T and ba^T. Discuss its interpretation.

Problem 3 (10 marks)

Suppose that we define a weighted inner product on $\mathbf{R}^2$ as follows: $\langle x, y \rangle = 4x_1y_1 + 5x_2y_2$. Note that this is a different from the standard inner product on $\mathbf{R}^2$

Given that $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, find:

(a) **[3 marks]** $||a||$

(b) **[3 marks]** $||b||$

(c) **[4 marks]** $|\langle a, b \rangle|^2$

Problem 4 (10 marks)

Compute the following:

(a) **[3 marks]** The cosine of the angle between $[47, 100, 0]$ and $[0, 0, 5]$

(b) **[3 marks]** The unit vector perpendicular to both $[-1, -2, -3]$ and $[5, 0, 2]$

(c) **[4 marks]** $|v \times w|$ given that v and w are two 3×1 vectors, their lengths are $|v| = 3$ and $|w| = 7$, and that the two are separated by an angle of $\pi/4$.

Problem 5 (6 marks)

Given $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $xy^T = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$, find y belonging to $\mathbf{R^3}$.

Problem 6 (10 marks)

Find the standard inverse of the following matrices. If it cannot be calculated, explain why. Show intermediate steps.

Hint: Recall the concept of the independence-dimension inequality.

(a) **[5 marks]** $a = \begin{bmatrix} 4 & 7 \\ 9 & 8 \end{bmatrix}$

(b) **[5 marks]** $b = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$

Problem 7 (6 marks)

Identify the greatest number of independent vectors among the following 6 vectors. State your reasoning.

$$a = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, d = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, e = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, f = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Problem 8 (8 marks)

Suppose you are given the following three vectors:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(a) [**3 marks**] State whether the set $\{ u, v, w \}$ is linearly independent. Justify.

(b) [**5 marks**] Formulate an equation with u, v, and w that makes the three vectors linearly dependent.

Problem 9 (10 marks)

For the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$

(a) [**2 marks**] Find its basis.

(b) [**4 marks**] Find a basis of its intersection with the xy plane.

(c) [**4 marks**] Find a basis for all the vectors that are perpendicular to the plane.

Problem 10 (5 marks)

You are given an $m \times n$ matrix A with rank r. For $Ax = b$, there exists a b for which the system does not have any solution. List the relationships between m, n, and r using inequality signs ($<, \leq, =, \geq, >$). If any two values are equal, show this with an equal sign.

Problem 11 (5 marks)

Suppose that a linear transformation T is outlined by the following matrix:

$$\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Determine if this matrix:

(a) [**2 marks**] Maps $\mathbf{R}^4$ to $\mathbf{R}^3$.

(b) [**3 marks**] Is a one-to-one mapping. Explain.

Problem 12 (8 marks)

For the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects points across the x-axis followed by a reflection across the line $y = x$,

(a) [**3 marks**] Find the transformation matrix.

(b) [**5 marks**] Show that this transformation is simply a rotation about the origin. Determine the angle of rotation.

— End of Assignment —