LUMS
A Not-for-Profit University

Department of Electrical Engineering
School of Science and Engineering

EE212 Mathematical Foundations for Machine Learning and Data Science

ASSIGNMENT 1 - Solution

Due Date: 23:55, Sunday, September 26, 2021. (Submit online on LMS)
Format: 12 problems, for a total of 100 marks
Instructions:

- You are allowed to collaborate with your peers but copying your colleague's solution is strictly prohibited. This is not a a group assignment. Each student must submit his/her own assignment, scanned in **a single PDF document.**
- Solve the assignment on blank A4 sheets and either scan the document using a scanner or use CamScanner proficiently.
- Upload the solved assignment on LMS in the "Assignments" tab under Assignment 01.
- Naming convention should be as follows: "Name_RollNumber_Assignment_1.pdf"
- Feel free to contact the instructor or the teaching assistants if you have any concerns.

- You represent the most competent individuals in the country, do not let plagiarism come in between your learning. In case any instance of plagiarism is detected, the disciplinary case will be dealt with according to the university's rules and regulations.

Problem 1 (8 marks)

Given the vectors below,

$$a = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Calculate the following.

(a) [**2 marks**] $2a + b$

(b) [**2 marks**] $7a - 4c$

(c) [**2 marks**] $(a \times b) \times c$, where '$\times$' denotes the cross product.

(d) [**2 marks**] $7a \cdot c$, where '$\cdot$' denotes the dot product.

Solution.

(a) $\begin{bmatrix} 12 \\ 12 \\ 6 \end{bmatrix}$.

(b) $\begin{bmatrix} 31 \\ 17 \\ 11 \end{bmatrix}$.

(c) $\begin{bmatrix} -6 \\ 30 \\ 24 \end{bmatrix}$.

(d) 49.

Problem 2 (14 marks)

For the vectors

$$a = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(a) [**2 marks**] Compute $a^T b$.

(b) [**2 marks**] Compute $b^T a$.

(c) [**2 marks**] Compute ab^T.

(d) [**2 marks**] Compute ba^T.

(e) [**3 marks**] Describe the relationship between $a^T b$ and $b^T a$. Discuss its interpretation.

(f) [**3 marks**] Describe the relationship between ab^T and ba^T. Discuss its interpretation.

Solution.

(a) $a^T b = -2a + 3b - 4c$.

(b) $b^T a = -2a + 3b - 4c$.

(c) $ab^T 4 = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$

(d) $ba^T = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$

(e) Since the inner product $a^T b$ is a real number, it equals its transpose. That is, $a^T b$=$(a^T b)^T = b^T (a^T)^T = b^T a$. The inner product tells us how much of one vector lies in the direction of the other.

(f) The outer product ab^T is an $n \times n$ matrix. Thus, $(ab^T)^T = (b^T)^T a^T = ba^T$.

Problem 3 (10 marks)

Suppose that we define a weighted inner product on $\mathbf{R}^2$ as follows: $\langle x, y \rangle = 4x_1 y_1 + 5x_2 y_2$. Note that this is a different from the standard inner product on $\mathbf{R}^2$

Given that $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, find:

(a) **[3 marks]** $||a||$

(b) **[3 marks]** $||b||$

(c) **[4 marks]** $|\langle a, b \rangle|^2$

Solution.

(a) Since $||a||^2 = \langle a, a \rangle = 9$, $||a|| = 3$.

(b) Since $||b||^2 = \langle b, b \rangle = 105$, $||b|| = \sqrt{105}$.

(c) $|\langle a, b \rangle|^2 = 15^2 = 225$.

Problem 4 (10 marks)

Compute the following:

(a) **[3 marks]** The cosine of the angle between $[47, 100, 0]$ and $[0, 0, 5]$

(b) **[3 marks]** The unit vector perpendicular to both $[-1, -2, -3]$ and $[5, 0, 2]$

(c) **[4 marks]** $|v \times w|$ given that v and w are two 3×1 vectors, their lengths are $|v| = 3$ and $|w| = 7$, and that the two are separated by an angle of $\pi/4$.

Solution.

(a) 0 , $\pi/2$.

(b) Find the normalized cross product.
The perpendicular unit vector could be $(-4/\sqrt{285}, -13/\sqrt{285}, 10/\sqrt{285})$ or $(4/\sqrt{285}, 13/\sqrt{285}, -10/\sqrt{285})$.

(c) $21\frac{\sqrt{2}}{2}$.

Problem 5 (6 marks)

Given $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $xy^T = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$, find y belonging to $\mathbf{R^3}$.

Solution.
Notice that the second row of the matrix is double the first row. Thus,
$xy^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$.

Problem 6 (10 marks)

Find the standard inverse of the following matrices. If it cannot be calculated, explain why. Show intermediate steps.
Hint: Recall the concept of the independence-dimension inequality.

(a) **[5 marks]** $a = \begin{bmatrix} 4 & 7 \\ 9 & 8 \end{bmatrix}$

(b) **[5 marks]** $b = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$

Solution.

(a) Inverse: $\begin{bmatrix} -8/31 & 7/31 \\ 9/31 & -4/31 \end{bmatrix}$.

(b) Inverse: A non-square matrix does not have a standard inverse.

Problem 7 (6 marks)

Identify the greatest number of independent vectors among the following 6 vectors. State your reasoning.

$$a = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, d = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, e = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, f = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution. Since $d = b + a$, $e = c + a$, and $f = c - b$, there are at most three independent vectors among these. Furthermore, applying row reduction to the matrix $[abc]$ gives three pivots, showing that a, b, and c are independent.

Problem 8 (8 marks)

Suppose you are given the following three vectors:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(a) [**3 marks**] State whether the set $\{ u, v, w \}$ is linearly independent. Justify.

(b) [**5 marks**] Formulate an equation with u, v, and w that makes the three vectors linearly dependent.

Solution.

(a) u and v are basic variables while w is free. Each nonzero value of v determines a nontrivial solution of the set's augmented matrix. Hence the vectors u, v, and w are linearly dependent.

(b) The augmented matrix of the three vectors can be written as $\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$.

Completely row reducing this matrix, we get $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus, $x_1 = 2x_2$, $x_2 = -x_3$, and x_3 is free. Picking an arbitrary value for x_3 such as 5, we get $x_1 = 10$ and $x_2 = -5$. A possible linear dependence equation becomes $10u - 5v + 5w = 0$. There are infinitely many solutions possible.

Problem 9 (10 marks)

For the plane $x - 2y + 3z = 0$ in $\mathbf{R^3}$

(a) [**2 marks**] Find its basis.

(b) [**4 marks**] Find a basis of its intersection with the xy plane.

(c) [**4 marks**] Find a basis for all the vectors that are perpendicular to the plane.

Solution.

(a) This plane is the nullspace of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The special solutions $a = [2, 1, 0]^T$ and $b = [-3, 0, 1]^T$ give a basis for the nullspace and thus for the plane.

(b) The intersection of this plane with the xy plane is a line. Since the first vector, a, lies in the xy plane, it must lie on the line and thus gives a basis for it.

(c) The vector $c = [1, -2, 3]^T$ is perpendicular to both vectors. Since the space of vectors perpendicular to a plane in R^3 is one-dimensional, it gives a basis.

Problem 10 (5 marks)

You are given an $m \times n$ matrix A with rank r. For $Ax = b$, there exists a b for which the system does not have any solution. List the relationships between m, n, and r using inequality signs ($<$, $\leq$, $=$, $\geq$, $>$). If any two values are equal, show this with an equal sign.

Solution. The rank of a matrix is always less than or equal to the number of rows and columns, so $r \leq m$ and $r \leq n$. Moreover, by the second statement, the column space is smaller than the space of possible output matrices, i.e. $r < m$.

Problem 11 (5 marks)

Suppose that a linear transformation T is outlined by the following matrix:

$$\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Determine if this matrix:

(a) [**2 marks**] Maps $\mathbf{R^4}$ to $\mathbf{R^3}$.

(b) [**3 marks**] Is a one-to-one mapping. Explain.

Solution.

(a) Since T happens to be in echelon form, we can see at once that it has a pivot position in each row. For each b in $\mathbf{R^3}$, the equation $Tx = b$ is consistent. In other words, the linear transformation T maps $\mathbf{R^4}$ (its domain) onto $\mathbf{R^3}$.

(b) Since the equation $Tx = b$ has a free variable (because there are four variables and only three basic variables), each b is the image of more than one x. That is, T is not one-to-one.

Problem 12 (8 marks)

For the transformation $T : \mathbf{R^2} \to \mathbf{R^2}$ that reflects points across the x-axis followed by a reflection across the line $y = x$,

(a) **[3 marks]** Find the transformation matrix.

(b) **[5 marks]** Show that this transformation is simply a rotation about the origin. Determine the angle of rotation.

Solution.

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(b) The transformation T maps a rotation about the origin through $\pi/2$ radians. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in R^2.

— End of Assignment —