Lahore University of Management Sciences
Syed Babar Ali School of Science and Engineering

EE212 Mathematical Foundations for Machine Learning and Data Science

Fall 2021

Mid Exam

Time Allowed: 90 minutes **Total Points**: 70

Instructions

1. We require you to solve the exam in a single time-slot of one hour and thirty minutes without any external or electronic assistance.
2. The exam is closed-book, closed-notes. One hand-written A4-sized formula sheets (two-sided) are allowed. Calculators are also allowed.
3. Try to identify the easiest way to solve a problem.
4. Clearly outline all your steps. Solutions with inadequate justifications and/or steps may not receive full credit.
5. We encourage you to solve the exam on A4 paper or electronic writing pads, use new sheet for each question and write sheet number on every sheet.
6. Your solution must be submitted on LMS before 2 PM. No submissions will be accepted beyond this time.
7. Exam consists of 9 problems worth a total of 70 points.

Problem 1. (**8 pts**)

A polynomial of degree $n-1$ is defined as

$$p(t) = c_1 + c_2\, t + c_3\, t^2 + \cdots + c_n\, t^{n-1},$$

where we note that the polynomial $p(t)$ is characterized by n number of coefficients $c_1, c_2, \ldots, c_n$, represented in the form of vector given by

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The derivative of the polynomial $p(t)$ with respect to t is defined as

$$p(t) = c_2 + 2c_3\, t + +3c_4\, t^2 \cdots + (n-1)c_n\, t^{n-2},$$

which is in fact a polynomial of degree $n-2$ and is characterized by $c_2, \ldots, c_n$, represented in the form

$$d = \begin{bmatrix} c_2 \\ 2c_3 \\ 3c_4, \\ \vdots \\ (n-1)c_n \end{bmatrix}$$

Design a system characterized by the matrix A such that $d = Ac$, that is, find a matrix A for which $d = Ac$.

Problem 2. (**6 pts**)

For a matrix $A \in \mathbf{R}^{m \times n}$ with columns denoted by $a_1, a_2, \ldots, a_n$, each column has unit norm and angle between a_i and a_j is 60^o. Find the Gram matrix $G = A^T A$.

Problem 3. (**12 pts**)

Show that the linear transformation using orthogonal matrix $A \in \mathbf{R}^{m \times n}$ preserves norm, angle, distance and inner product, that is, show the following for two vectors $x, y \in \mathbf{R}^n$

(a) $(Ax)^T(Ay) = x^T y$

(b) $\|Ax\| = \|x\|$

(c) $\|Ax - Ay\| = \|x - y\|$

(d) $\angle(Ax, Ay) = \angle(x, y)$

Problem 4. (**4 pts**)

Suppose $AX = AY$ for matrices A, X and Y. We can conclude immediately that $X = Y$ if these are scalars and $A \neq 0$. What condition on A is needed in the matrix case to conclude that $X = Y$? Provide justification to support your answer.

Problem 5. **(10 pts)**

For a matrix $A \in \mathbf{R}^{m \times n}$ with SVD $A = U\Sigma V^T$, we have the following information available.

(i) Rank of a matrix is 2.

(ii) Two non-zero eigenvalues of AA^T are 3 and 1.

(iii) Third left singular vector u_3 is given by $\frac{1}{\sqrt{3}}\begin{bmatrix} 0 & -1 & -1 \end{bmatrix}^T$

(iv) $Av_1 = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$ and $Av_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$, where v_1 and v_2 are the first and second columns of the matrix V respectively.

(a) Determine the size of the matrix A.

(b) Determine the matrix U.

Problem 6. (**8 pts**)

Each of a set of m dengue patients can exhibit any number of a set of n symptoms. This data can be expressed in the form of a matrix $S \in \mathbf{m} \times \mathbf{n}$ where

$$S_{i,j} = \begin{cases} 1 & \text{patient } i \text{ exhibits symptom } j \\ 0 & \text{patient } i \text{ does not exhibit symptom } j \end{cases}$$

Provide simple descriptions of the following expressions (where $\mathbf{1}$ is a vector of all ones).

(a) $S^T \mathbf{1}$

(b) $S\mathbf{1}$

(c) $S^T S$

(d) $S S^T$

Problem 7. (6 pts)

We define a matrix $A \in \mathbf{R}^{n \times n}$ to positive semi-definite (PSD) if

$$x^T A\, x \geq 0, \quad \forall x \in \mathbf{R}^n$$

and positive definite (PD) if

$$x^T A\, x > 0 \quad \forall x \in \mathbf{R}^n,\, x \neq 0.$$

Given this definition, how do you interpret geometrically the matrix-vector product using PSD or PD matrices?

Problem 8. (8 pts)

Consider two $m \times n$ matrices A and B. Suppose that for $j = 1, 2, \ldots n$, the j-th column of A is a linear combination of the first j columns of B. How do we express this as a matrix equation? Choose one of the matrix equations below and justify your choice.

(a) $A = BJ$ for some lower triangular matrix J

(b) $A = FB$ for some lower triangular matrix F.

(c) $A = BH$ for some upper triangular matrix H

(d) $A = GB$ for some upper triangular matrix G

Problem 9. (**8 pts**)

A set of K different types of crude oil are blended (mixed) together in proportions $\theta_1, \ldots, \theta_K$. These numbers sum to one; they must also be nonnegative, but we will ignore that requirement here. Associated with crude oil type k is an n-vector c_k that gives its concentration of n different constituents, such as specific hydrocarbons. Find a set of linear equations on the blending coefficients, $A\theta = b$, that expresses the requirement that the blended crude oil achieves a target set of constituent concentrations, given by the n-vector c^{tar}. (Include the condition that $\theta_1 + \ldots + \theta_K = 1$ in your equations.)