Mathematical Foundations for Machine Learning and Data Science

Eigenvalue Decomposition (EVD)

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Outline

• Eigenvalue Decomposition
  • Eigenvectors, Eigenvalue overview
  • Formulation
  • Interpretation
Eigenvalue Decomposition (EVD)

Eigenvectors and Eigenvalues:
For square matrices, *eigenvectors* and *eigenvalues* are vectors and numbers represent the *eigen-decomposition* of a matrix; analyzes the structure of this matrix.

For a matrix $A \in \mathbb{R}^{n \times n}$

a vector $q \in \mathbb{R}^n$, $q \neq 0$, is called an eigenvector of $A$ if

$$Aq = \lambda q$$

- $\lambda$ is referred to as an eigenvalue of $A$ associated with the eigenvector $q$
Eigenvalue Decomposition (EVD)

Linear transformation interpretation:

\[ q \in \mathbb{R}^n \quad \xrightarrow{A} \quad \lambda q \in \mathbb{R}^n \quad \text{Only scaling} \]

Graphically:

\[ Aq = \lambda q \]
Eigenvalue Decomposition (EVD)

**Eigenvectors and Eigenvalues:**

How to compute eigenvalues?

\[ A \mathbf{q} = \lambda \mathbf{q} \quad \Rightarrow \quad A\mathbf{q} - \lambda \mathbf{q} = 0 \]

\[ p(\lambda) = \det(A\mathbf{q} - \lambda \mathbf{q}) = 0 \quad \text{Characteristic polynomial; degree } n \]

\[ n \text{ eigenvalues} \quad \text{Eigenspectrum: set of eigenvalues} \]

\[ n \text{ eigenvectors} \quad \text{Eigenspace: span of eigenvectors} \]
Eigenvalue Decomposition (EVD)

Eigen vectors and eigenvalues:

\[ A q = \lambda q \]

\( n \) eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \)

\( n \) eigenvectors \( q_1, q_2, \ldots, q_n \in \mathbb{R}^n \)

\[ A q_1 = \lambda_1 q_1, \quad A q_2 = \lambda_2 q_2, \quad \ldots \]

\[ A Q = Q \Lambda \]

\[ A = Q \Lambda Q^{-1} \]

Eigen-decomposition
Eigenvalue Decomposition

What does eigenvector and eigenvalues reveal about $A$?

**Linear Transformation**

$x \xrightarrow{} A \xrightarrow{} A x$

**Linear Transformation Interpretation in terms of Eigen-decomposition of the Matrix**

$x \in \mathbb{R}^n \xrightarrow{} Q^{-1} \xrightarrow{} \Lambda \xrightarrow{} Q \xrightarrow{} A x$

Transformation (Change of Basis) Scaling (In new Basis) Inverse Transformation (Change of Basis)
Eigenvalue Decomposition

Linear Transformation Interpretation in terms of Eigen-decomposition of the Matrix - Visualization

Linear Transformation

Transformation (Change of Basis)

Scaling (In new Basis)

Inverse Transformation (Change of Basis)
Eigenvalue Decomposition

Eigen-decomposition of the Matrix - Example

\[ \lambda_1 = 2.0 \]
\[ \lambda_2 = 0.5 \]
\[ \det(A) = 1.0 \]
Eigenvalue Decomposition

Eigen-decomposition of the Matrix - Example

\[ \lambda_1 = 1.0 \]
\[ \lambda_2 = 1.0 \]
\[ \det(A) = 1.0 \]
Eigenvalue Decomposition

**Eigen-decomposition of the Matrix - Example**

\[
\begin{align*}
\lambda_1 &= 0.0 \\
\lambda_2 &= 2.0 \\
\text{det}(A) &= 0.0
\end{align*}
\]
Eigenvalue Decomposition

Determinant in terms of Eigenvalues

- Determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is given by the product of eigenvalues, that is,

$$\det(A) = \det(Q \Lambda Q^{-1}) = \det(Q) \det(A) \det(Q^{-1})$$

$$\det(A) = \det(Q \Lambda Q^{-1}) = \det(Q) \left( \prod_{i=1}^{n} \lambda_i \right) \frac{1}{\det(Q)}$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

- If one of the eigenvalues is zero, determinant of the matrix is zero.

  - I encourage you to connect this with the interpretation of determinant and (eigenvectors, eigenvalues).
Eigenvalue Decomposition (EVD)

**EVD of Inverse Matrix:**

- Matrix inverse $A^{-1}$ has same eigenvectors but eigenvalues are given by inverse of the eigenvalues of the original matrix $A$.

- This can be shown in multiple ways. Let’s use eigenvalue equation to show this.

For a matrix $A \in \mathbb{R}^{n \times n}$

$$Aq = \lambda q$$  

**Eigenvalue Equation**

- Assuming $A$ is invertible, that is, all eigenvalues are non-zero.

$$A^{-1}Aq = \lambda A^{-1}q$$

$$\frac{1}{\lambda}q = A^{-1}q$$

- This implies $q$ is an eigenvector of $A$ with associated eigenvalue $\frac{1}{\lambda}$. 
Eigenvalue Decomposition

**Power of a matrix**

\[ AA = A^2 = Q\Lambda Q^{-1} \]

\[ A^2 = Q\Lambda^2 Q^{-1} \]

\[ A^n = Q\Lambda^n Q^{-1} \]

- I encourage you to connect this with the interpretation of linear transformation using EVD of a matrix.
Eigenvalue Decomposition

Zero eigenvalues; Columns of $A$ are not linearly independent

- If one of the eigenvalues is zero and $q$ is an associated eigenvector.

$$A q = \lambda q = 0$$

- It simply follows from the definition of linear independence that the columns are not linearly independent since $Aq$ represents the linear combination of columns of $A$. 
Eigenvalue Decomposition

**Eigenvalues of a Symmetric Matrix**

- (Spectral Theorem) For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, there exists an orthonormal basis of the corresponding vectors space consisting of eigenvectors of $A$ and each eigenvalue is real.

\[
A = Q \Lambda Q^{-1} \quad \text{Since } A = A^T, \text{ we have } (Q^{-1} = Q^T)
\]

\[
A^T = (Q^{-1})^T \Lambda Q^T
\]

- Summary: For a symmetric matrix, eigenvectors are orthonormal and eigenvalues are real.