LUMS
A Not-for-Profit University

Department of Electrical Engineering
School of Science and Engineering

EE212 Mathematical Foundations for Machine Learning and Data Science

TUTORIAL 5 – SOLUTIONS

Tutorial 5-1

The Pakistan hockey has 2 matches scheduled in the qualifying round for next Olympics. Pakistan team has a 0.4 probability of not losing the first game, and a 0.6 probability of not losing the second game. If it does not lose, the team has a 50% chance of a win, and a 50% chance of a tie, independently of all other factors. Pakistan team will receive 2 points for a win, 1 for a tie, and a 0 for a loss. Let X be the number of points the Pakistan team earns during the qualifying round. Find the PMF of X.

Solution: This requires enumeration of the possibilities and straightforward computation:

$$\begin{aligned}
\mathbf{P}(X = 0) &= 0.6 \cdot 0.4 = 0.24, \\
\mathbf{P}(X = 1) &= 0.4 \cdot 0.5 \cdot 0.4 + 0.6 \cdot 0.5 \cdot 0.6 = 0.26 \\
\mathbf{P}(X = 2) &= 0.4 \cdot 0.5 \cdot 0.4 + 0.6 \cdot 0.5 \cdot 0.6 + 0.4 \cdot 0.5 \cdot 0.6 \cdot 0.5 = 0.32 \\
\mathbf{P}(X = 3) &= 0.4 \cdot 0.5 \cdot 0.6 \cdot 0.5 + 0.4 \cdot 0.5 \cdot 0.6 \cdot 0.5 = 0.12 \\
\mathbf{P}(X = 4) &= 0.4 \cdot 0.5 \cdot 0.6 \cdot 0.5 = 0.06 \\
\mathbf{P}(X > 4) &= 0
\end{aligned}$$

Tutorial 5-2

Let X and Y be normal random variables with mean 0 and 1, respectively, and variances 1 and 4, respectively.

(a) Find $\mathbf{P}(X \leq 1.5)$ and $\mathbf{P}(X \leq -1)$.

(b) What is the PDF of $Z = (Y - 1)/2$.

(c) Find $\mathbf{P}(-1 \leq Y \leq 1)$.

Solution:

(a) X is a standard normal, so we have $\mathbf{P}(X \leq 1.5) \leq \phi(1.5) = 0.9332$. Also $\mathbf{P}(X \leq -1) = 1 - \phi(1) = 1 - 0.8413 = 0.1587$.

(b) By subtracting from Y its mean and dividing by standard deviation, we obtain the standard normal.

(c) We have $\mathbf{P}(-1 \leq Y \leq 1) = \mathbf{P}(-1 \leq (Y-1)/2 \leq 0) = \mathbf{P}(-1 \leq Z \leq 0) = \mathbf{P}(0 \leq Z \leq 1) = \phi(1) - \phi(0) = 0.8413 - 0.5 = 0.3413$.

Tutorial 5-3

Find the PDF, the mean, and the variance of the random variable X with CDF

$$F_X(x) = \begin{cases} 1 - \frac{a^3}{x^3} & \text{if } x \geq a, \\ 0 & \text{if } x < a, \end{cases}$$

where a is a positive constant.

Solution: We have

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} 3a^3x^{-4} & \text{if } x \geq a, \\ 0 & \text{if } x < a, \end{cases}$$

Also

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^{\infty} x \cdot 3a^3x^{-4} dx = 3a^3 \int_a^{\infty} x^{-3} dx = 3a^3 \left(-\frac{1}{2}x^{-2}\right)\Big|_a^{\infty} = \frac{3a}{2}.$$

Finally, we have

$$\mathbf{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_a^{\infty} x^2 \cdot 3a^3x^{-4} dx = 3a^3 \int_a^{\infty} x^{-2} dx = 3a^3 \left(-x^{-1}\right)\Big|_a^{\infty} = 3a^2,$$

so the variance is

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = 3a^2 - \left(\frac{3a}{2}\right)^2 = \frac{3a^2}{4}.$$

Tutorial 5-4

For an exponential random variable X with $\lambda = 1$, find all moments of X.