LUMS
A Not-for-Profit University

Department of Electrical Engineering
School of Science and Engineering

EE212 Mathematical Foundations for Machine Learning and Data Science

ASSIGNMENT 2

Due Date: 13:30, Thursday, October 20, 2022.
Format: 6 problems, for a total of 100 marks
Instructions:

- You are allowed to collaborate with your peers but copying your colleague's solution is strictly prohibited. This is not a group assignment. Each student must submit his/her own assignment.
- Solve the assignment on blank A4 sheets and staple them before submitting.
- Submit in-class or in the dropbox labeled EE-212 outside instructor's office.
- Write your name and roll no. on the first page.
- Feel free to contact the instructor or the teaching assistants if you have any concerns.

- You represent the most competent individuals in the country, do not let plagiarism come in between your learning. In case any instance of plagiarism is detected, the disciplinary case will be dealt with according to the university's rules and regulations.

Problem 1 (20 marks)

For the following matrices, find eigenvalues and the corresponding eigenvectors. Show the steps of the solution.

(a) **[10 marks]** $\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) **[10 marks]** $\begin{bmatrix} 5 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 5 & 3 \end{bmatrix}$

Problem 2 (15 marks)

Consider SVD of a certain matrix A as

$$A = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{U}} \Sigma \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}}_{V^T}$$

In $Ax = y$, following information is known: $U^T y = [5 \quad 3]^T and \;\; Vx = [-3 \quad 1.67]^T$.
$Find \;\; \Sigma \;\; matrix.$

Problem 3 (10 marks)

Let A be an m x n matrix such that the nullspace of A is a plane in $\mathbb{R}^3$. The column space is spanned by vector v in $\mathbb{R}^5$.

(a) [**4 marks**] Find the size of matrix A.

(b) [**2 marks**] What is the number of linearly independent columns of A?

(c) [**4 marks**] Find rank and nullity of A.

Problem 4 (10 marks)

For a square matrix A, the characteristic polynomial is given as:

$$\det(A - \lambda I) - = (\lambda - 1)^3(\lambda - 2)^2(\lambda - 6)^4(\lambda - 3)$$

Find the rank of A.

Problem 5 (30 marks)

Consider the following matrix A_1 where the rows represent the data points:

$$\begin{bmatrix} 10 & 0 & 10 \\ -10 & 0 & -10 \\ 30 & 20 & -20 \end{bmatrix}$$

Due to some noise, the actual observation of A_1 is given by:

$$C_1 = A_1 + noise = \begin{bmatrix} 9.9999 & 0.0003 & 10.0001 \\ -9.9991 & 0.0001 & -10.9972 \\ 30.0002 & 20.0004 & -20.0012 \end{bmatrix}$$

Apparently, the noise has increased the rank of the original matrix A_1 from 2 to 3. The underlying matrix A_1 is of low rank as compared to C_1 and thus provides us the information on which features are redundant in the data.
Now suppose another matrix C_2 has been obtained similar to C_1 and has an underlying low-rank matrix A_2. C_1 is not the same as C_2. You have been provided with the following information about the SVD of $C_2 = U\Sigma V^T$.

1. C_2 has 3 singular values.
2. One of the singular values is 0.0008 and the corresponding left-singular vector is $[-0.7064 \quad -0.7078 \quad -0.0002]^T$ and the corresponding right singular vector is $[0.3480 \quad -0.08704 \quad -0.3483]^T$.
3. Two common eigenvalues of $C_2^T C_2$ and $C_2 \, C_2^T$ are 3.8407 and 17.1596.
4. The eigenvector of $C_2 \, C_2^T$ corresponding to the eigenvalue 3.8407 is $[0.7038 \quad -0.7023 \quad -0.1068]^T$. The eigenvalue of $C_2 \, C_2^T$ corresponding to the eigenvalue 17.1596 is partially known to be $[unknown \;\; unknown \quad -0.9943]^T$.

5. The eigenvector of $C_2^T C_2$ corresponding to the eigenvalue 17.1596 is $[0.7566 \ \ -0.4802 \ \ 0.4439]^T$. The eigenvalue of $C_2^T C_2$ corresponding to the eigenvalue 3.8407 is partially known to be $[unknown \ \ -0.1090 \ \ unknown]^T$.

(a) [**3 marks**] Find size of matrix C_2.

(b) [**3 marks**] Determine Σ matrix.

(c) [**8 marks**] Determine the U matrix.

(d) [**8 marks**] Determine the V matrix.

(e) [**8 marks**] Find the best approximation of A_2.

Problem 6 (15 marks)

An n x n matrix A is called skew-symmetric if A^T = -A, i.e., its transpose is its negative. (A symmetric matrix satisfies $A^T = A$.)

(a) [**3 marks**] Find all 2 x 2 skew-symmetric matrices.

(b) [**3 marks**] Explain why the diagonal entries of a skew-symmetric matrix must be zero.

(c) [**4 marks**] Show that for a skew-symmetric matrix A, and any n-vector x, $(Ax) \perp x$. This means that Ax and x are orthogonal. *Hint.* First show that for any n x n matrix A and n-vector x, $x^T(Ax) = \sum_{i,j=1}^{n} A_{ij}x_ix_j$.

(d) [**5 marks**] Now suppose A is any matrix for which $(Ax) \perp x$ for any n-vector x. Show that A must be skew-symmetric. *Hint*. You might find the formula:

$$(e_i + e_j)^T(A(e_i + e_j)) = A_{ii} + A_{jj} + A_{ij} + A_{ji}$$

valid for any n x n matrix A, useful. For $i = j$, this reduces to $e_i^T\ (Ae_i) = A_{ii}$.