First-Order Circuits - Lecture Notes

Monday, 11 November 2019 12:38 PM

First Order Circuits:

A circuit with only one energy storage element (capacitor or Inductor) is referred to as 'First Order Circuit'.

Why: The network equations describing the circuit are first order differential equations. In other words, current through or voltage across any element in the circuit is a solution of first order differential equation.

There are two types of first-order circuits: RL circuit and RC circuit

Note: If there are multiple energy storage elements in the circuit such that they can be reduced to a single equivalent energy storage element, the circuit is first-order. For example, multiple capacitors/inductors in series/parallel combination.

Concept of Switching and Equivalent Model of Elements

For convenience, we operate the switch in circuits at t=0.
We define three times

- 0^- - Time instant before operating the switch
- 0^+ - Time instant after operating the switch
- ∞ - Infinity (final state of the circuit)

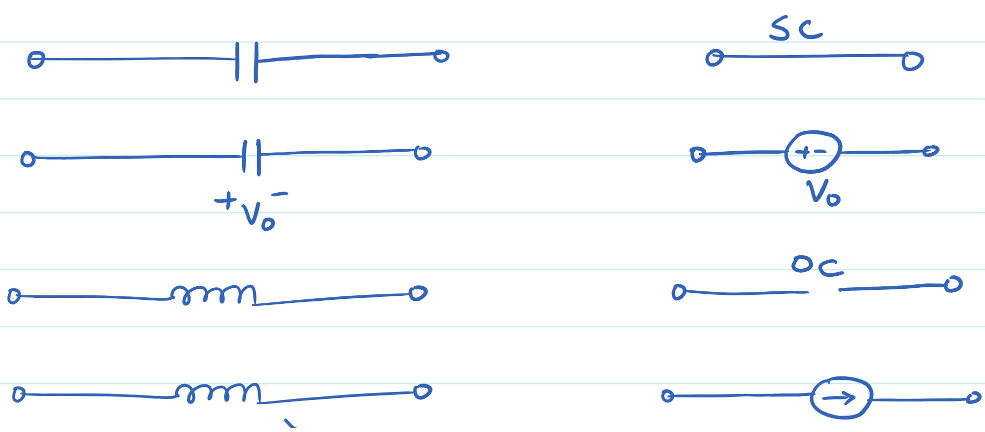
Energy storage elements serve as memory elements in the circuit and therefore these should be analysed at the time of switching.

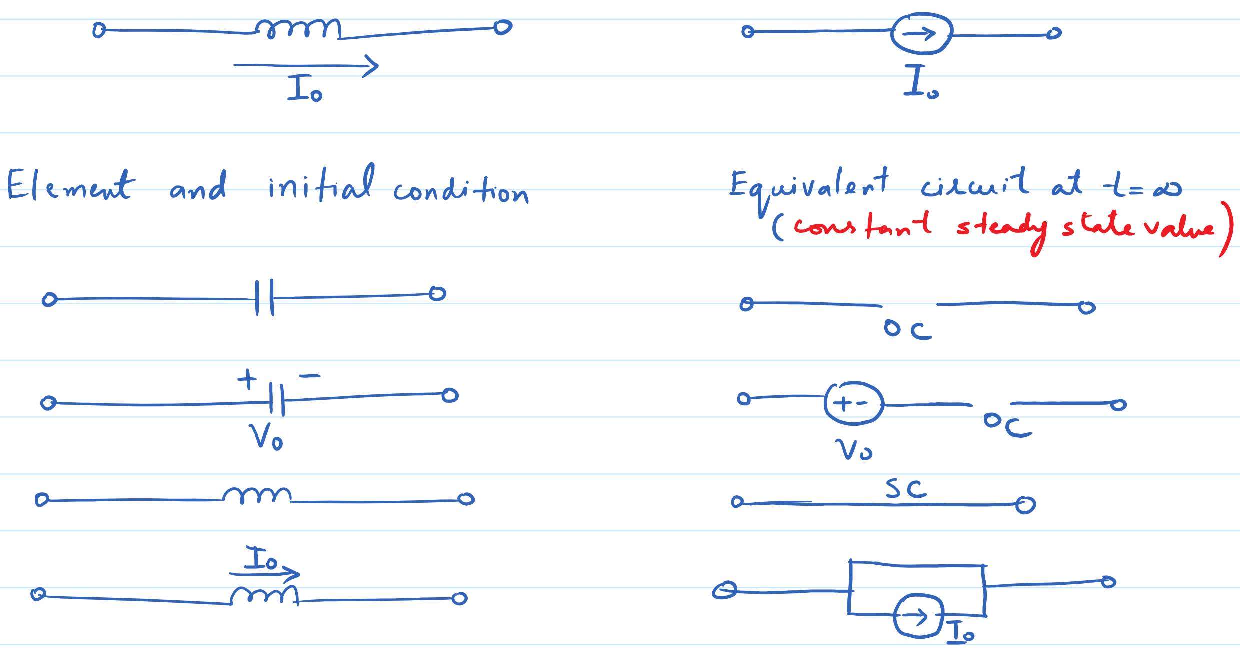
To analyse the behaviour of the energy storage elements , we recall the characteristics of energy storage elements.

- Capacitor does not allow instantaneous change in voltage
  - *Why:* $i = C\frac{dV}{dt}$. To change the voltage instantaenously, we require infinite current.
  - Except when the capacitor is connected directly to a voltage source. In such case, infinite current is supplied by the source to charge the capacitor and consequently change the voltage.
- Inductor does not allow instantaneous change in current
  - *Why:* $v = L\frac{di}{dt}$. To change the current instantaenously, we require to apply infinite voltage across inductor.
  - Except when the inductor is connected directly to a current source. In such case, infinite voltage builds up across the inductor to change the flux and consequently change the current.
- If the voltage across capacitor is constant (not varying with time), the current through the capacitor is zero. Zero current -> Open-circuit
- If the current through inductor is constant (not varying with time), the voltage across inductor is zero. Zero voltage -> Short-circuit

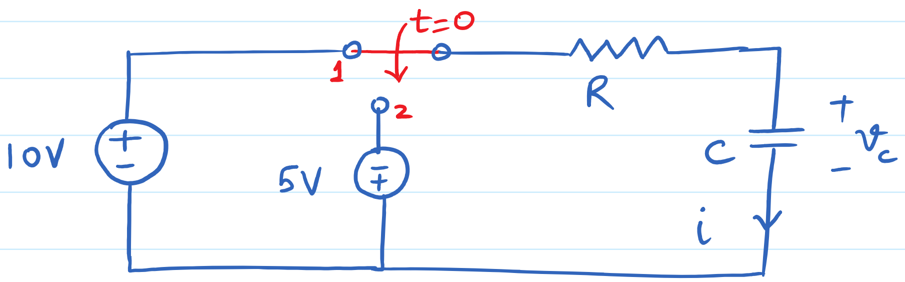
Taking into account these considerations, we note the following behaviour of elements at the time of switching and at t = ∞.

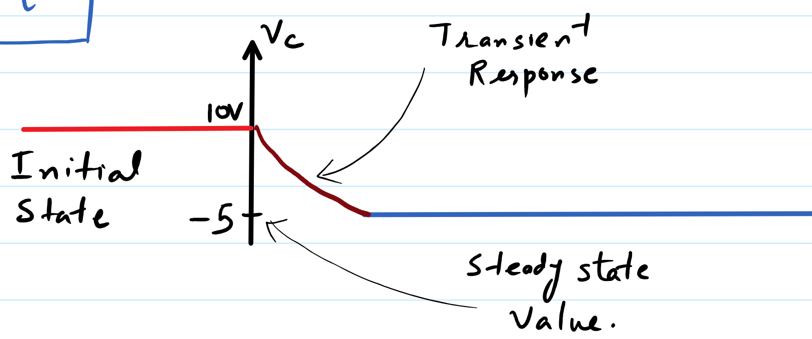
Element and Condition at $t=0^-$ Equivalent circuit Model at $t=0^+$





Now we quickly analyse one circuit without going into much detail. Consider RC series circuit connected to a voltage source of 10V. At t=0, the circuit is disconnected from 10V source and connected to -5V source using a switch. Assume that the circuit has been connected to 10V source for very long time such that the capacitor is fully charged.





Q: How does the voltage across capacitor change?
A: Capacitor voltage changes from 10V to -5V. *Simple!*

Q: How does the current through change?
A: Current will be negative for some time and becomes zero (once capacitor voltage reaches -5V). *Simple!*

But

- How much time does the voltage take to decrease from 10V to -5V?
- How does the voltage decrease from 10V to -5 V (linear, oscillatory, exponential)?
- How do we analyse the more complicated circuit?

You will be able to answer these questions by the time we finish our analysis of First-order circuits.

Using this example, we can also define "initial value", "transient response" and "steady state value". For example, the initial value (or state) of the capacitor voltage is 10V. The steady state value, by definition the value at t=∞, is -5V. The transient response refers to the response of the circuit from the initial state to the steady state value. These have been indicated on the plot above.

Solution of First-Order Differential Equation:

To facilitate the students not taking Engineering Modelling, we quickly review the solution of first order differential equation.

We consider a nonhomogeneous first-order differential equation of the form:

$$\frac{dy(t)}{dt} + P\,y(t) = Q(t)$$

- *For any circuit analysis problem*, P is a positive constant that is determined by the parameters (resistors and capacitor/inductor).
- $y(t)$ - current through or volatge across any element of the circuit, also called response of the circuit.
- $Q(t)$ - forcing function or excitation. When $Q(t)$ is zero, the equation is referred to as homogeneous differential equation.

We have the following solution of the differential equation

$$y(t) = \underbrace{Ke^{-Pt}}_{\text{Complementary solution } y_c(t)} + \underbrace{e^{-Pt}\int Q(t)\,e^{Pt}\,dt}_{\text{Particular integral } y_p(t)}$$

$y_c(t)$ - Complementary solution - Also referred to as transient response as it decays to zero as time progress (P is positive constant)
$y_p(t)$ - Particular Integral and is also referred to as steady stare response since $y_c(t)$ decays to zero over time.

Special Case - $Q(t) = 0$

$$y(t) = Ke^{-Pt}$$

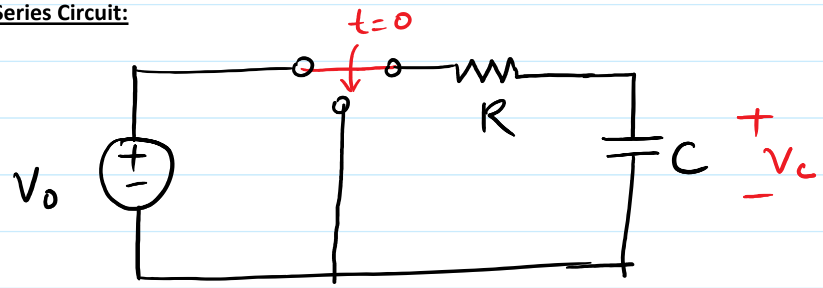
Special Case - $Q(t) = Q\ (constant)$

$$y(t) = \frac{Q}{P} + Ke^{-Pt}$$

Circuits without sources (Excitation):

Now we start analysing circuits without any sources, also called, source-free circuits as the circuit does not contain any sources after operating the switch.

RC Series Circuit:



Problem 1: Determine the voltage across capacitor for $t \geq 0$.

Initial conditions (state) of the capacitor (Analyse circuit at t=0$^-$)

Capacitor is fully charged. $v_C(0^-) = v_0 = v_C(0^+)$

After the switch is operated

$$C\frac{\mathrm{d}v_C}{\mathrm{d}t} + \frac{v_C}{R} = 0\,, \qquad \frac{\mathrm{d}v_C}{\mathrm{d}t} + \frac{v_C}{CR} = 0,$$

$$v_C = k\mathrm{e}^{-\frac{t}{RC}} = k\mathrm{e}^{-\frac{t}{\tau}} \quad t \geq 0$$

$\tau = RC$ is referred to as time constant.

$$v_C = v_0\mathrm{e}^{-\frac{t}{\tau}} \quad t \geq 0$$

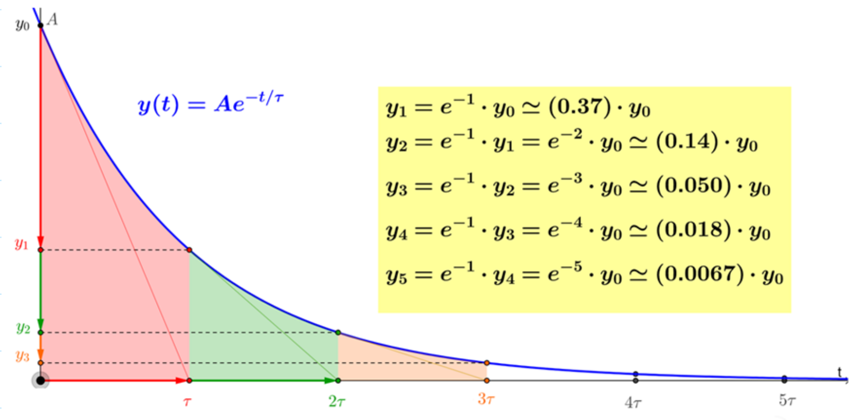
(Using initial conditions)

Interpretation of the response: Once the source is removed, the capacitor voltage decays to zero from v_0. The rate of decay is governed by the time constant of the circuit.

Larger the value of time constant, more time the voltage takes to decay to zero. Time constant serves as a measure of the amount of memory of the circuit as it quantifies the retention time of the voltage across capacitor.

In the limiting sense, when $R=\infty$, an ideal capacitor retains the voltage forever. Practical capacitor decays slowly to zero due to the parasitic resistance.

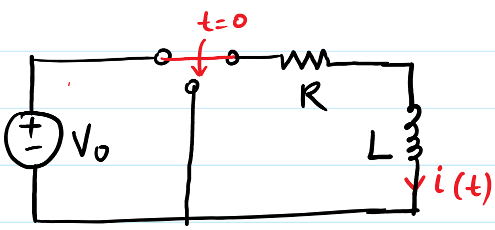
Let's also quickly review the plot of $A\mathrm{e}^{-\frac{t}{\tau}}$.



Source: https://www.geogebra.org/m/dYTScKrP

Problem 2: Determine the current through the capacitor for $t \geq 0$. Do it yourself, please :-)!

RL Series Circuit:



Problem 1: Determine the current through the inductor for $t \geq 0$.

Initial conditions (state) of the inductor (Analyse circuit at t=0$^-$)

Inductor is carrying constant current. $\underline{i(0^-) = I_0 = i(0^+)}$

After the switch is operated

$$L\frac{\mathrm{d}\,i}{\mathrm{d}t} + Ri = 0\,, \qquad \frac{\mathrm{d}\,i}{\mathrm{d}t} + \frac{R}{L}i = 0,$$

$$i = k\mathrm{e}^{-\frac{R\,t}{L}} = k\mathrm{e}^{-\frac{t}{\tau}} \quad t \geq 0$$

$\tau = L/R$ is referred to as time constant.

$$i(t) = I_0 e^{-\frac{t}{\tau}} \quad t \geq 0$$ (Using initial conditions)

Interpretation of the response: Once the source is removed, the inductor current decays to zero from I_0. The rate of decay is governed by the time constant of the circuit.

Larger the value of time constant, more time the current takes to decay to zero. Time constant serves as a measure of the amount of memory of the circuit as it quantifies the retention time of the current through inductor.

First Order Circuits with DC Sources:

* Now, we look at circuits with DC sources.

* DC sources $\longrightarrow$ excitation function $Q(t) = Q$
  i.e., time independent

* For $Q(t) = Q$; Solution of $\frac{dy(t)}{dt} + P\,y(t) = Q(t)$ is

$$y(t) = e^{-Pt} Q \int e^{Pt} dt + K e^{-Pt}$$

$$\Rightarrow \quad y(t) = \frac{Q}{P} + K e^{-Pt} \quad \text{—} \quad (01)$$

For convenience and easy interpretation; we reformulate as

$$\boxed{y(t) = K_1 + K_2\, e^{-t/\tau}}$$

- This is equivalent to formulation in (01)

* $K_1 = \lim_{t \to \infty} y(t) = y(\infty)$
* $K_2 = y(0) - K_1$
* τ — Circuit Time Constant

$y(\infty)$ – steady state value

$y(0)$ – value at 0^+

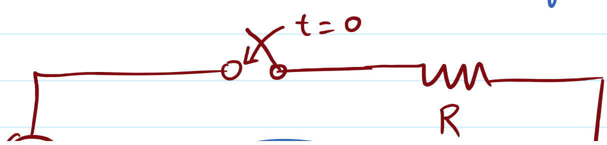
$\Rightarrow \quad \tau = R_{eq} C \quad$ or $\quad \tau = \frac{L}{R_{eq}}$

Storage Element : capacitor inductor

R_{eq} : Equivalent Resistance across terminals of C or L.

Example :

* Series RC with constant voltage source



Problem 1: We want to find $i(t)$

$\underline{0^-}$:

* $i(0^-) = 0$
* $V_c(0^-) = 0$

$\underline{0^+}$:

* $i(0^+) \; \frac{V_0}{R}$
* $V_c(0^+) = 0$

* Equation for circuit when switch is closed.

$$iR + \frac{1}{C}\int i\,dt = V_0 \Rightarrow \frac{di}{dt} + \frac{1}{RC} i = 0$$

$$\Rightarrow i(t) = K e^{-t/RC}$$

$$i(t) = \begin{cases} \frac{V_0}{R} e^{-t/RC} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Problem 2: Find $v_c(t)$.

We can determine $v_c(t)$ as integral of $i(t)$ as

$$v_c(t) = \frac{1}{C}\int i(t)\,dt$$

Alternatively, we can formulate differential equation

Equation: $C\frac{dv_c}{dt} + \frac{v_c - v_0}{R} = 0 \Rightarrow \frac{dv_c}{dt} + \frac{v_c}{RC} = \frac{V_0}{RC}$

Solution: $v_c(t) = \frac{Q}{P} + K e^{-pt}$ $\quad Q = \frac{V_0}{RC}$

$P = \frac{1}{RC}$

$$\Rightarrow v_c(t) = V_0 + K e^{-t/RC}$$

$v_c(0) = 0 = V_0 + K \Rightarrow K = -V_0$

$$\boxed{v_c(t) = V_0\left(1 - e^{-t/RC}\right)}$$

In both of these problems; solution is of the form

$$K_1 + K_2 e^{-t/\tau} \qquad \tau = RC$$

Problem 01:

$$K_1 = i(\infty) = 0$$

$$K_2 = i(0) - K_1 = \frac{V_0}{R}$$

* 0^+

Circuit at 0^+

Problem 02:

$$K_1 = v_c(\infty) = V_0$$

$$K_2 = v_c(0) - K_1 = -V_0$$

Conclusion:

* For any First-order circuit (switching at $t=0$) current through any element or voltage across any element is given by

$$y(t) = K_1 + K_2 e^{-t/\tau} \qquad t \geq 0$$

* Let's learn how to determine K_1, K_2 and τ for a given circuit

Determine K_1: Recall $K_1 = y(\infty)$

- Replace C (or L) with equivalent model (See Fig 5.3 textbook)

After replacement; we have a circuit without C (or L).

- Determine $y(\infty) = K_1$

Determine K_2:

- Determine voltage across C or current through L at $t=0^-$ ★
- Replace C or L with equivalent model at $t=0^+$ (switch operated)
  (See Fig 5.2, textbook)
- Determine $y(0^+)$ and $K_1 = y(0^+) - K_2$

Determine τ

Recall $\tau \rightarrow$ circuit time constant

* Determine R_{eq} across C or L for a circuit after operating the switch.

* $\tau = R_{eq} C$ or $\tau = \frac{L}{R_{eq}}$.