Second-Order Circuits -Lecture Notes

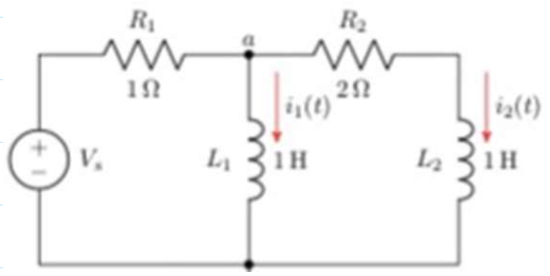
Monday, 11 November 2019 4:16 PM

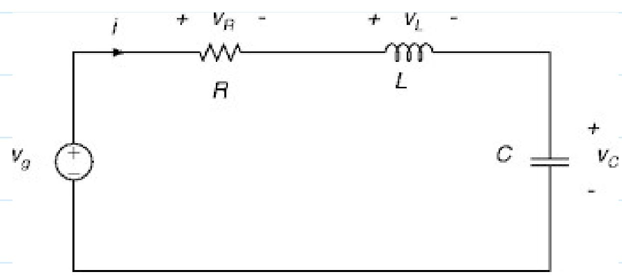
Second-Order Circuits:

A circuit with two energy storage elements (capacitors and/or Inductors) is referred to as 'Second-Order Circuit'.

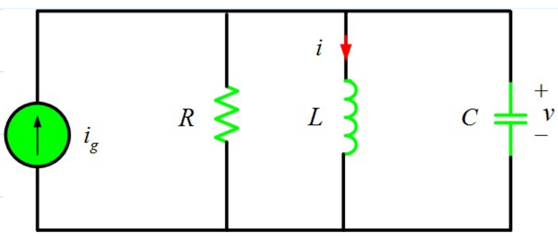
Why: The network equations describing the circuit are second order differential equations. In other words, current through or voltage across any element in the circuit is a solution of second order differential equation.

Following are the examples of second-order circuits.





Series RLC



Parallel RLC

Analysis of Second-Order Circuits

Analysis of second-order circuits requires us to solve second-order differential equation.

We therefore quickly review the solution of second-order differential equation.

Nonhomogeneous linear constant coefficient (LCC) second-order differential equation of the form:

$$\frac{d^2y(t)}{dt^2} + a_1\,\frac{dy(t)}{dt} + a_2\,y(t) = Q(t)$$

- *For any circuit analysis problem*, a_1 and a_2 are non-negtaive constants that are determined by the parameters (resistors and capacitor/inductor) of the circuit.
- $y(t)$ - current through or volatge across any element of the circuit, also called response of the circuit.
- $Q(t)$ - forcing function or excitation. When $Q(t)$ is zero, the equation is referred to as homogeneous differential equation.

We have the following solution of the differential equation

$$y(t) = y_c(t) \qquad + \qquad y_p(t)$$

Complementary solution Particular integral

$y_c(t)$ - Complementary solution
$y_p(t)$ - Particular Integral

Complementary Solution

Complementary solution is a solution of associated homogeneous differential equation. To find the complementary solution, we form the **characteristic equation**, also referred to as *auxiliary equation*, using the coefficients in the differential equation as

$$s^2 + a_1 s + a_2 = 0$$

The solution (roots) of the characteristic equation is given by

$$s_1, s_2 = -\frac{a_1}{2} \frac{\pm\sqrt{a_1^2 - 4a_2}}{2}$$

Based on the values of the constants, the roots can be real or complex. We therefore categorize the solution into 4 types, depending on the nature of the roots.

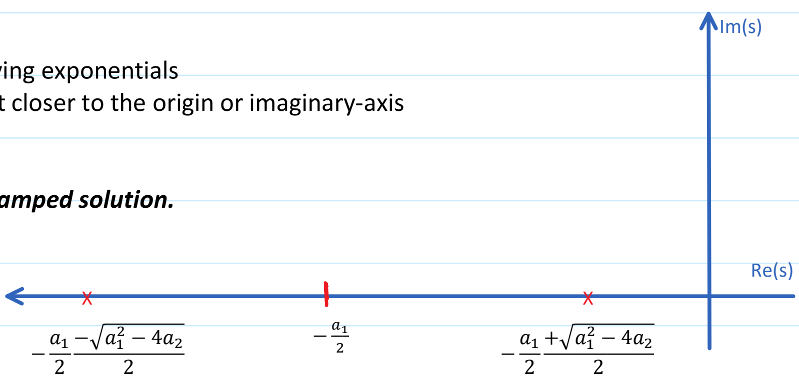
<u>Case 1: $a_1^2 - 4a_2 > 0$</u>

- The roots are real and distinct, indicated in the s-plane on the right
- The roots are centred around $-\frac{a_1}{2}$
- The solution is of the form

$$y_c(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

- Interpretation: The solution is a sum of two decaying exponentials
    - The exponential due to the root closer to the origin or imaginary-axis decays slowly over time

- Such solution or response is referred to as ***over-damped solution.***



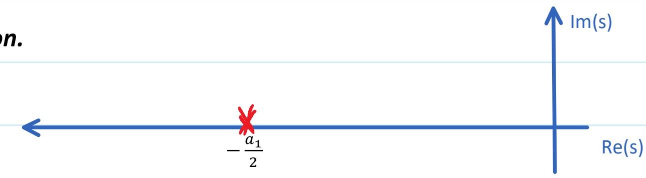
Case 2: $a_1^2 - 4a_2 = 0$

- The roots are real and equal, located at $-\frac{a_1}{2}$
- The solution is of the form

$$y_c(t) = k_1 \mathrm{e}^{s_1 t} + k_2\ t\ \mathrm{e}^{s_1 t} = \ \mathrm{e}^{s_1 t}\ (k_1 + k_2\ t)$$

- Interpretation: The solution is a sum of decaying exponentials times a linear function of time

- Such solution or response is referred to as ***critically-damped solution.***



Case 3: $a_1^2 - 4a_2 < 0$

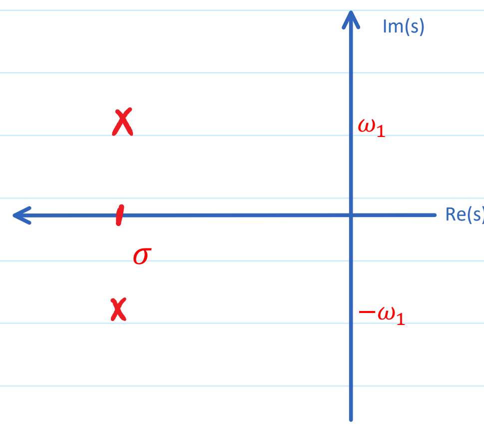
- The roots are complex and appear as complex conjugate pair

- Let's reformulate the roots in complex form as

$$s_1, s_2 = -\frac{a_1}{2} \frac{\pm \boldsymbol{j}\sqrt{4a_2 - a_1^2}}{2}$$

- The solution is of the form

$$y_c(t) = k_1 \mathrm{e}^{s_1 t} + k_2\ \mathrm{e}^{s_2 t}$$



- By introducing the notation as $\sigma = -\frac{a}{2}$ and $\omega_1 = \frac{\sqrt{4a_2 - a_1^2}}{2}$, the solution can be reformulated as

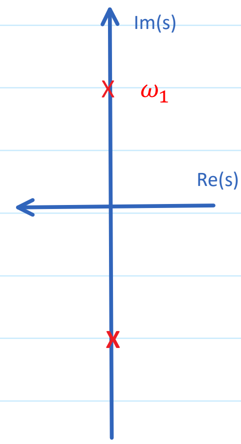
$$y_c(t) = k_1 \mathrm{e}^{(\sigma + j\,\omega_1)t} + k_2\ \mathrm{e}^{(\sigma - j\,\omega_1)t}\ =\ \mathrm{e}^{\sigma t}\,(k_3 \cos \omega_1 t +\ k_4 \sin \omega_1 t)$$

- Interpretation: The solution is a product of decaying exponential and oscillatory function.
- The decay rate is determined by the real part of the root and the frequency of oscillation is determined by the imaginary part of the root.

- Such solution or response is referred to as ***under-damped solution.***

Case 4: $a_1 = 0$

- The roots are complex (purely imaginary) and appear as complex conjugate pair
- Let's reformulate the roots in complex form as



$$s_1, s_2 = \frac{\pm \boldsymbol{j}\sqrt{4a_2}}{2}$$

- The solution is of the form

$$y_c(t) = (k_1 \cos \omega_1 t + k_2 \sin \omega_1 t)$$

- $\omega_1 = \dfrac{\sqrt{4a_2}}{2}$

- Interpretation: The solution is oscillatory in nature
- Such solution or response is referred to as ***un-damped solution,*** as there is no damping (decaying exponential in the solution).

Particular Solution or Particular Integral

To find particular solution, we use a technique, referred to as **"method of undetermined coefficients".**

In this method, a trial function is assumed to be a particular solution. A trial function is chosen based on the given form or mathematical representation of the excitation function *Q(t).* We assign constants (undetermined coefficients) to the trial function and substitute the solution in the differential equation to determine the value of undetermined coefficients.

Since *Q(t)* represents voltage or current source (excitation) , it can only take few mathematical forms such as constant (DC), exponential, ramp, sinusoidal etc. We use the following table to choose trial function as particular solution

$Q(t)$	$y_p(t)$
K (constant)	A
$K\,t$	$At + B$
$K\,t^2$	$At^2 + Bt + C$
$K e^{-\alpha t}$	$A e^{-\alpha t}$
$K \cos \omega t$ $K \sin \omega t$	$A \cos \omega t + B \sin \omega t$

Remark : * If $Q(t)$ is a sum of different factors, $y_p(t)$ is also a sum of corresponding factors.

* If any factor in $y_p(t)$ is also a part of $y_c(t)$ we multiply $y_p(t)$ with 't'.

* the constants A, B.. are determined by substituting $y_p(t)$ in the differential equation.

Analysis of Second-Order Circuits

We assume now we are capable to carry out the following.

- Write network equations using KCL and KVL
- Evaluation of Initial Conditions
- Determine solution of second-order differential equation

This is all we need to analyse second-order circuits. The most important step in the analysis of second-order or higher-order circuits is the formulation of differential equation in terms of variable of interest.

You should choose the loop variables or nodal voltages while writing network equations such that the equations are formulated in terms of variable of interest. Once you have the network equations, you should learn to manipulate the equations such that you have a differential equation.

Once you have the differential equation, you can determine complementary solution and particular integral (if required) and use initial conditions to determine the constants of integration.

Caution: The constants of integration *must* be determined using the complete solution, that is, the sum of complementary and particular solutions.

The problems covered in-class have been uploaded on the course page.

I encourage you to review the sections 6.1, 6.2 and 6.3 of the textbook. See Table 6.1 along with the examples in the book