EE240 – Circuits I

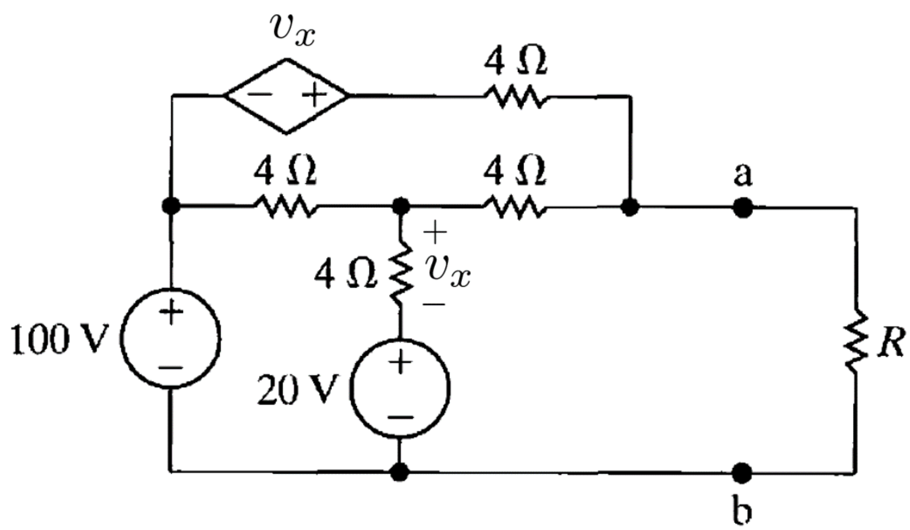
Final Examination (Fall 2020)

December 21, 2020 12:30 pm–03:30 pm

INSTRUCTIONS:

- We require you to solve the exam in a single time-slot of two hours and thirty minutes without any external or electronic assistance.
- We encourage you to solve the exam on A4 paper, use new sheet for each question and write sheet number on every sheet.
- The exam is closed book and notes. You are allowed to have two A4 sheet with you with hand-written notes on both sides. Calculators can be used.
- For the sake of completeness, we require you to write the following statement on your first page of submission: *I commit myself to uphold the highest standards of (academic) integrity.*
- Reading time: 10 minutes
- Writing time: 2 hours and 30 minutes
- Submission time: 20 minutes
- The exam consists of 6 problems worth a total of 90 points.

Problem 1. (**15 pts**) The variable resistor R in the circuit given below is adjusted until it absorbs the maximum power from the circuit.



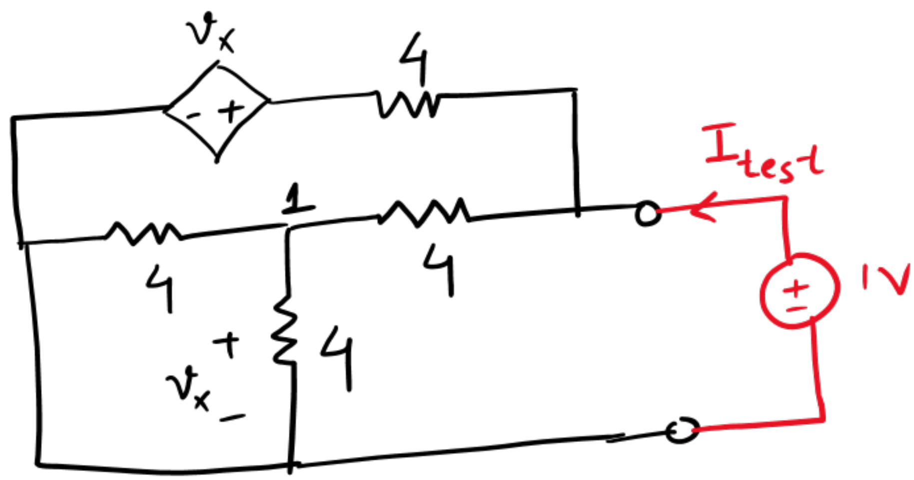
(a) (**6 pts**) Calculate the value of R for maximum power.

solution:
For maximum power, $R = R_{\text{th}}$. Switching off sources, connecting 1V voltage source across terminals a-b and applying KCL at node 1 yields

$$\frac{v_x}{4} + \frac{v_x}{4} + \frac{v_x - 1}{4} \Rightarrow v_x = \frac{1}{3}V$$

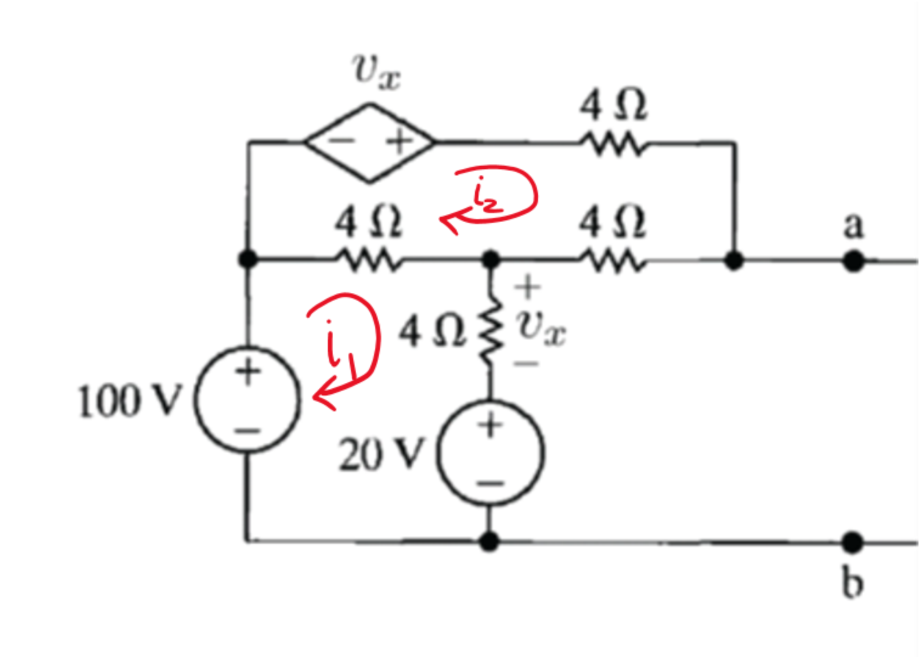
$$I_{\text{test}} = 2\frac{1 - \frac{1}{3}}{4} = \frac{1}{3}$$

$$R_{\text{th}} = \frac{1}{I_{\text{test}}} = 3\,\Omega.$$



(b) (**6 pts**) Determine the maximum power absorbed by R.

solution:



We determine V_{th} in order to find the power. Considering the loop currents indicated, we have $v_x = 4i_1$ Using KVL

Loop 1 equation:

$$8i_1 - 4i_2 = 80$$

Loop 2 equation:

$$-4i_1 + 12i_2 - v_x = 0 \Rightarrow 12i_2 = 8i_1$$

These two equations yield

$$i_1 = 15\,A, \quad i_2 = 10\,A$$

Using these currents, we determine V_{th} as

$$V_{\text{th}} = 4i_1 + 4i_2 + 20 = 120\,V$$

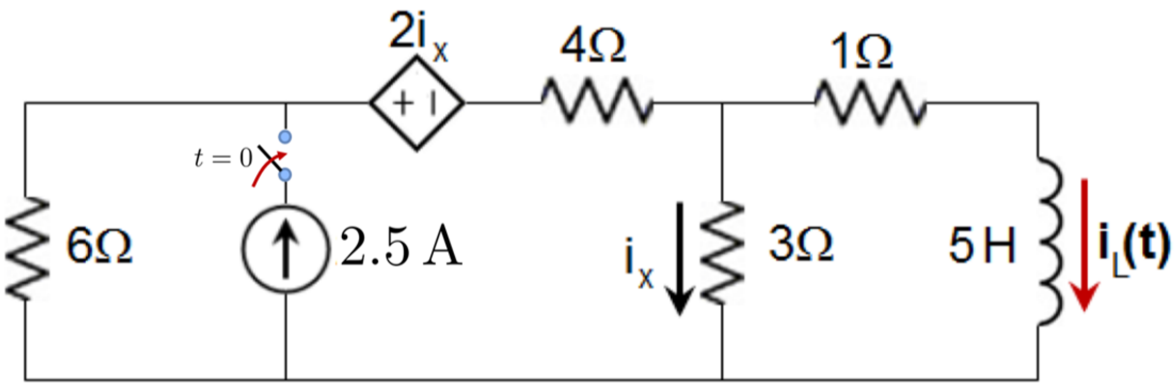
Now we have Thevenin equivalent circuit across terminals $a - b$. Power across R_{th} can be determine as: $P = \frac{60^2}{R_{\text{th}}} = 1200\,W$

(c) (**3 pts**) Determine the Norton equivalent of the circuit at terminals a-b.

solution:

Source transformation of Thevenin equivalent yields Norton equivalent.

Problem 2. (**15 pts**) The circuit given below is in steady state with switch in open state. The switch is closed at $t = 0$.



(a) (**2 pts**) Determine the current $i_L(t)$ at $t = 0^-$.

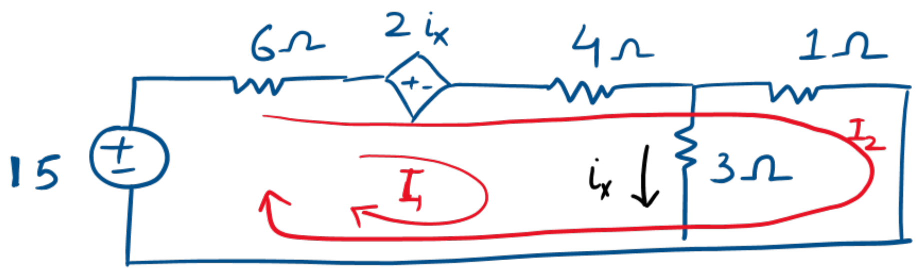
solution: $i_L(t) = 0$ at $t = 0^-$

(b) (**1 pts**) Determine the current $i_L(t)$ at $t = 0^+$.

solution: $i_L(t) = 0$ at $t = 0^+$

(c) (**5 pts**) Determine the current $i_L(t)$ at $t = \infty$

solution:
Switch is closed and inductor is short-circuit, we have the following circuit:



Applying KVL yields

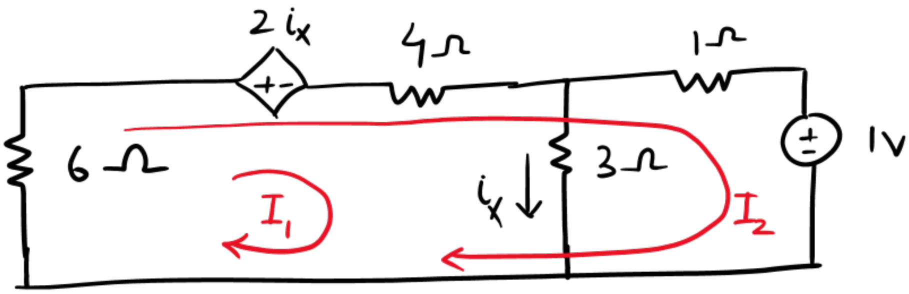
$$15I_1 + 10I_2 = 15 \Rightarrow 3I_1 + 2I_2 = 3$$

$$12I_1 + 11I_2 = 15$$

Solving for I_2, we get $i_L(\infty) = I_2 = 1\,A$.

(d) (**7 pts**) Using the results of the previous parts, or otherwise, determine the current $i_L(t)$ for all times and plot it.

solution:
We need to find time-constant or equivalent resistance across inductor using Thevenin theorem approach. By switching off sources and replacing inductor with 1V test source, we obtain the following circuit:



Applying KVL yields

$$15I_1 + 10I_2 = 0 \Rightarrow 3I_1 + 2I_2 = 0$$

$$12I_1 + 11I_2 = -1$$

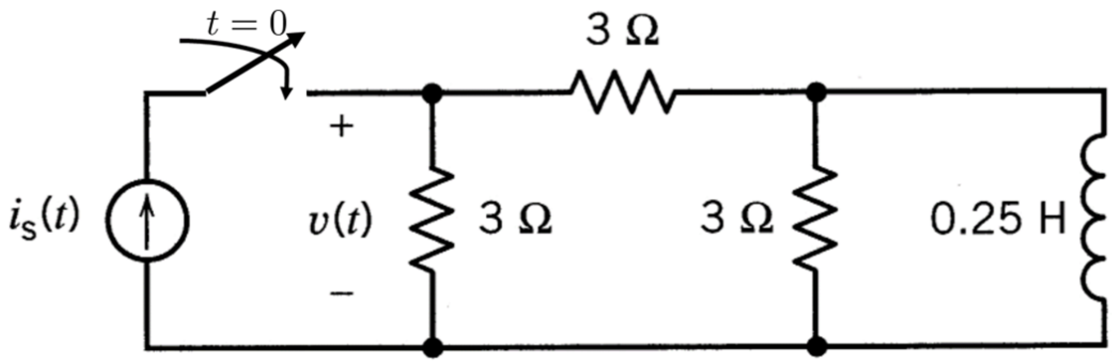
Solving for I_2, we get $I_2 = -\frac{1}{3}\,A$. R_{eq} across inductor is given by $R_{\text{eq}} = \frac{1}{I_{\text{test}}} = -\frac{1}{I_2} = 3\Omega$.
Now, $i_L(t)$ for $t \geq 0$ is given by

$$i_L(t) = K_1 + K_2 e^{-t/\tau}, \quad t \geq 0$$

$K_1 = i_L(\infty) = 1$, $K_2 = i_L(0) - i_L(\infty) = -1$, $\tau = L/R_{\text{eq}} = 5/3\,\text{s}$.

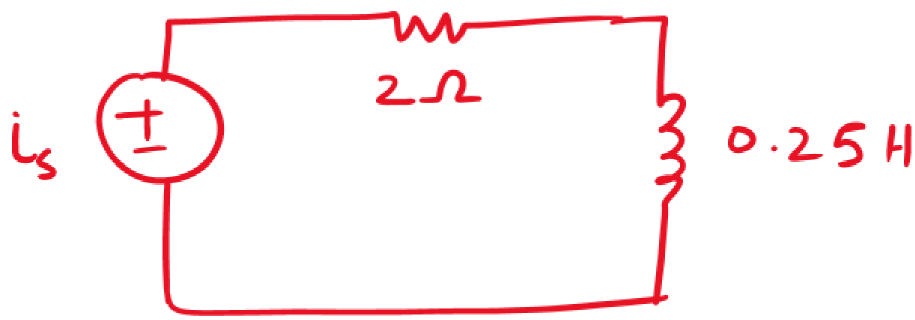
$$i_L(t) = 1 - 1e^{-3t/5}, \quad t \geq 0.$$

Problem 3. (**15 pts**) Consider the circuit given below. The circuit is in steady state and the the switch is closed at $t = 0$.



(a) (**4 pts**) For a circuit for $t > 0$, form an equivalent series RL circuit containing one source, one resistor and one inductor. Ignore $v(t)$ indicated in the circuit. (Hint: You may use source transformation or Thevenin equivalent approach.)

solution:
Using source transformation, we obtain a circuit with voltage source $i_s(t)$, resistor of resistance 2Ω and inductor.



(b) (**8 pts**) For $i_s(t) = e^{-t}\cos t$, determine the current through the inductor for $t \geq 0$. Ignore $v(t)$ indicated in the circuit.

solution:
Let $i(t)$ be the current through inductor. We first note that $i(0^+) = 0$. We have a series RL circuit. Applying KVL yields

$$\frac{di}{dt} + 8i = 4e^{-t}\cos t.$$

The solution is a sum of following two solution

$$i_c(t) = K_1 e^{-8t}$$

$$i_p(t) = Ae^{-t}\cos t + Be^{-t}\sin t$$

Solving for A and B, we get $A = 0.56$ and $B = 0.08$.

$$i(t) = K_1 e^{-8t} + 0.56e^{-t}\cos t + 0.08e^{-t}\sin t, \quad t \geq 0$$

Using initial conditions, $K_1 = -0.56$.

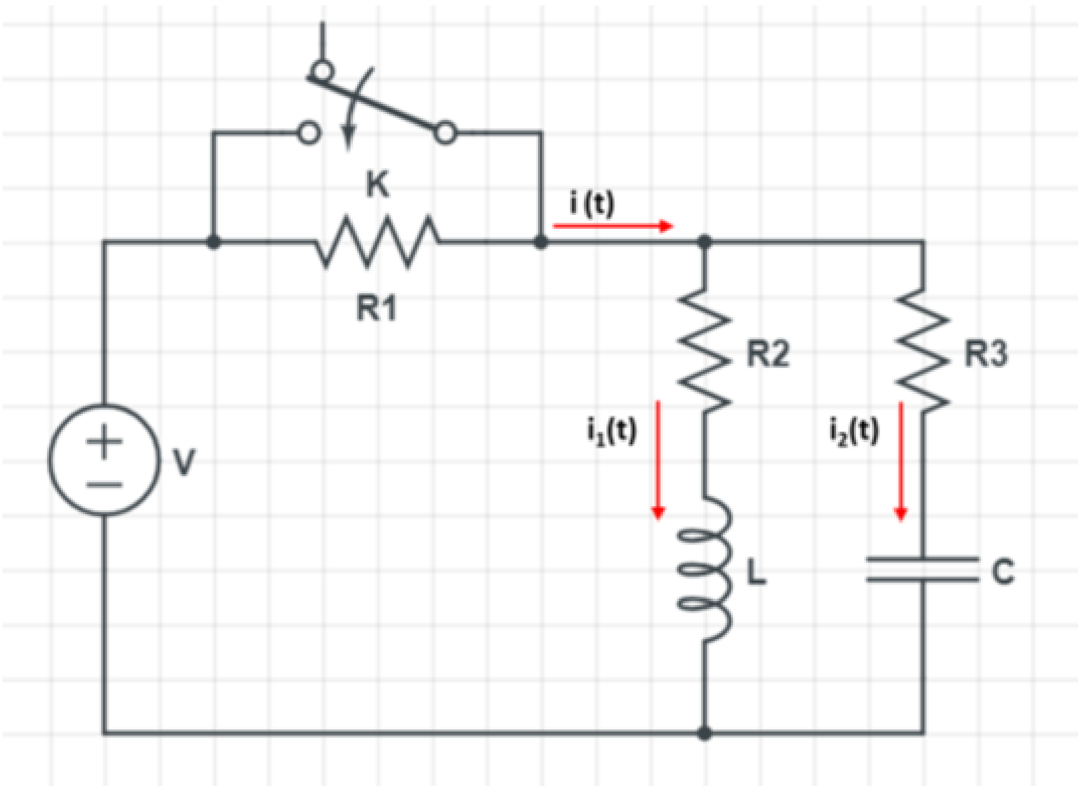
(c) (**3 pts**) Using the result of previous part or otherwise, determine the voltage $v(t)$ indicated in the circuit for $t \geq 0$.

solution:

$$v(t) = (0.25)\frac{di}{dt} + 3\left(i(t) + \frac{1}{3}(0.25)\frac{di}{dt}\right) = 3i(t) + \frac{1}{3}\frac{di}{dt}\ V.$$

Problem 4. (**10 pts**) In the circuit below, we have $R_2 = R_3 = 20\Omega$. We assume that the steady state is reached with switch K open. At time $t = 0$, the switch is closed. Determine values of resistance R_1, inductance L, capacitance C and voltage V of the voltage source using the following information.

- $i_1(0^+) = 2\,\text{A}$
- $i_2(0^+) = 1\,\text{A}$
- $\frac{di_1}{dt}(0^+) = 40\,\text{A/s}$
- $\frac{di_2}{dt}(0^+) = -\frac{1}{2}\,\text{A/s}$



solution:
Noting

$$2 = i_1(0^+) = i_1(0^-) = \frac{V}{R_1 + R_2}, \quad (1)$$

$$1 = i_2(0^+) = \frac{V - v_c}{R_3} = \frac{VR_1}{R_3(R_1 + R_2)}, \quad (2)$$

$$40 = \frac{di_1}{dt}(0^+) = \frac{VR_1}{L(R_1 + R_2)}, \quad (3)$$

$$-\frac{1}{2} = \frac{di_2}{dt}(0^+) = -\frac{i_2(0^+)}{R_3 C}. \quad (4)$$

First two equations yield

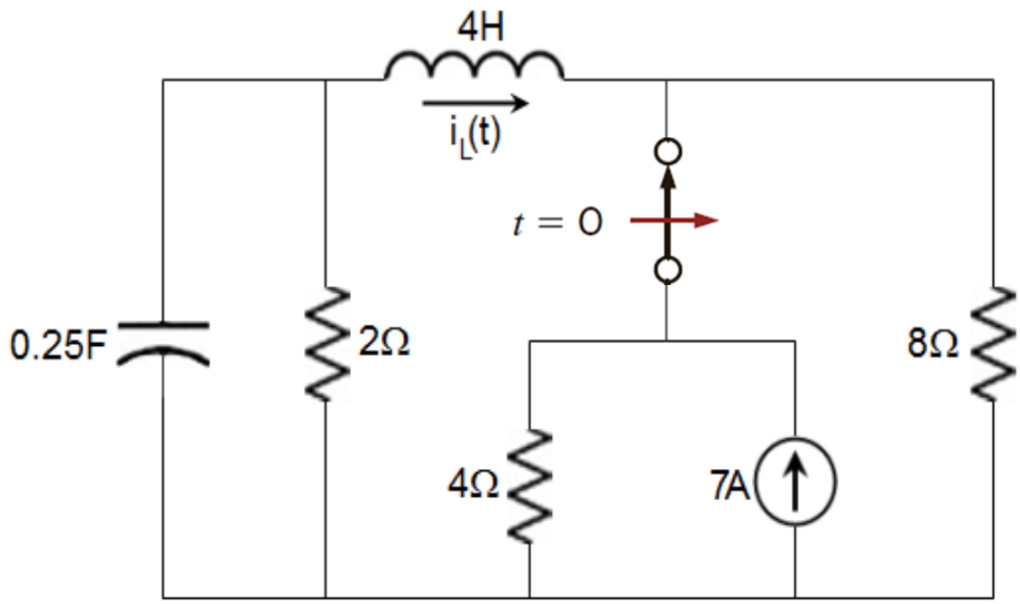
$$R_1 = 2R_3 = 10\Omega, \quad V = 60\,\text{Volts}$$

Using (3) and (4), we obtain

$$L = 0.5\,H, \quad C = 0.1\,F$$

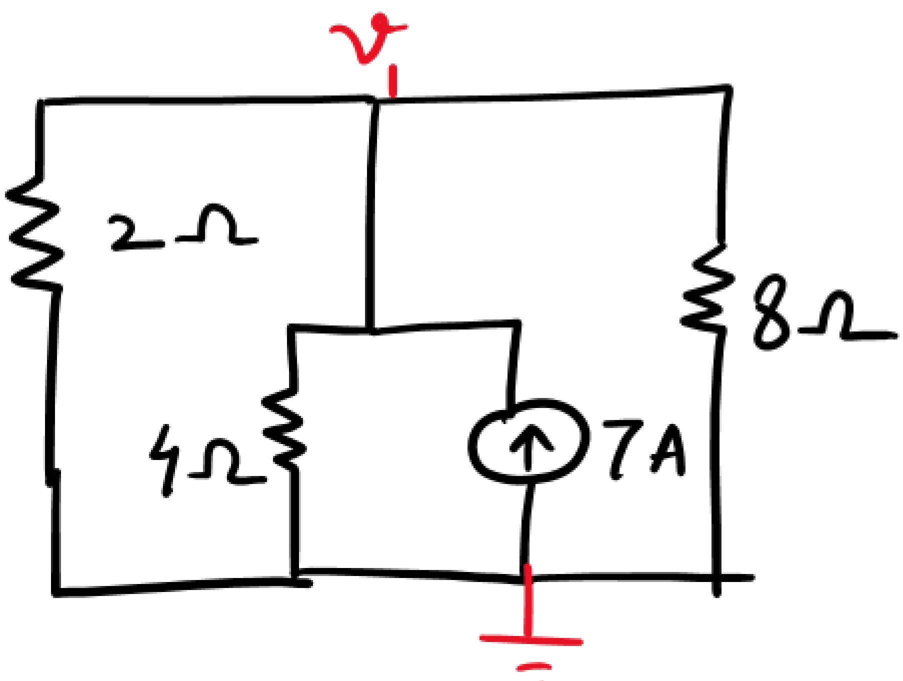
Problem 5. (**20 pts**)

Consider the circuit shown below. The circuit is in steady state with switch in closed state. The switch is opened at $t = 0$.



(a) (**6 pts**) Determine $i_L(t)$ and $\frac{di_L}{dt}$ at $t = 0^+$.

solution:
We have the following circuit at $t = 0^-$.



KCL yields

$$\frac{v_1}{2} + \frac{v_1}{4} + \frac{v_1}{8} = 7 \Rightarrow v_1 = 8V$$

At $t = 0^-$, we have

$$i_L(0^-) = -\frac{v_1}{2} = -4A, \quad v_c(0^-) = v_1 = 8V$$

At $t = 0^+$, we have

$$i_L(0^+) = i_L(0^-) = -4A$$

In order to find $\frac{di_L}{dt}(0^+)$, we note that we have

$$4\frac{di_L}{dt} + 8i_L(t) = v_c(t) \Rightarrow \frac{di_L}{dt}(0^+) = 10\,\text{A/s}$$

(b) (**5 pts**) Formulate the second-order differential equation describing the current $i_L(t)$ for $t \geq 0$.

solution:

We first note

$$4\frac{di_L}{dt} + 8i_L(t) = v_c(t)$$

KCL yields

$$i_L + \frac{v_c}{2} + 0.25\frac{dv_c}{dt} = 0$$

Manipulation of these two equations yield

$$i_L + 2\frac{di_L}{dt} + 4i_L(t) + \frac{d^2i_L}{dt^2} + 2\frac{di_L}{dt} = 0 \Rightarrow \frac{d^2i_L}{dt^2} + 4\frac{di_L}{dt} + 5i_L(t) = 0$$

(c) (**2 pts**) Determine the damping ratio ζ and natural frequency ω_n of the circuit after the switch is closed.

solution:
Comparing the characteristic equation of the differential equation given by

$$s^2 + 4s + 5 = 0$$

with the standard form

$$s^2 + 2\zeta w_n s + w_n^2 = 0,$$

we obtain $w_n = \sqrt{5}$ and $\zeta = \frac{2}{\sqrt{5}}$.

(d) (**7 pts**) Determine and plot (with labels) $i_L(t)$ for all times.

solution:
The roots of the characteristic equation associated with the differential equation has the roots

$$s = -2 \pm j,$$

and therefore the solution is undernamed and has the form

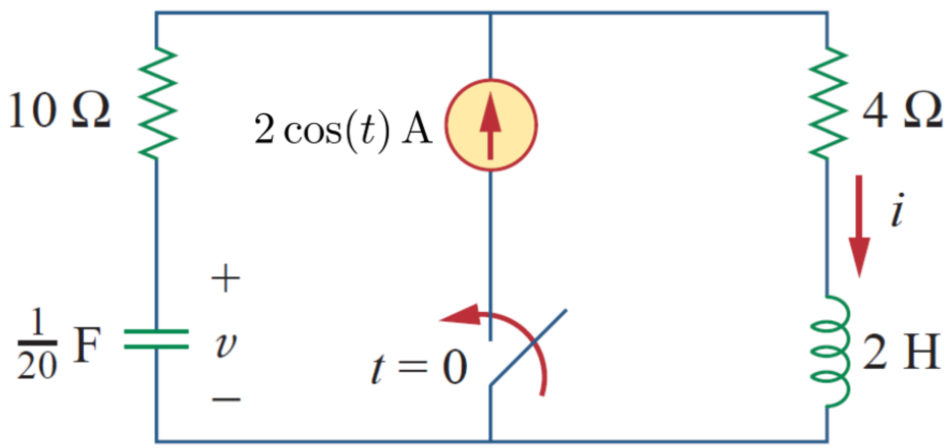
$$i_L(t) = e^{-2t}\big(K_1 \cos t + K_2 \sin t\big)$$

Using the initial conditions evaluated in part (a), $K_1 = -4$, and $K_2 - 2K_1 = 10, \Rightarrow K_2 = 2$.
Overall

$$i_L(t) = \begin{cases} e^{-2t}\big(-4\cos t + 2\sin t\big) & t \geq 0 \\ -4 & t < 0 \end{cases}$$

Problem 6. (**15 pts**)

The switch in the following circuit is closed at $t = 0$.



(a) (**5 pts**) Formulate the second order differential equation describing the voltage $v(t)$ after the switch is closed.

solution:
We first note

$$v + 12\frac{dv}{dt} = 4i + 2\frac{di}{dt}$$

KCL yields

$$i = 2\cos t - \frac{1}{20}\frac{dv}{dt}$$

Manipulation yields

$$v + 12\frac{dv}{dt} = 8\cos t - \frac{1}{5}\frac{dv}{dt} - 4\sin t - 110\frac{d^2v}{dt^2}$$

$$\frac{1}{10}\frac{d^2v}{dt^2} + \frac{1}{5}\frac{dv}{dt} + v + \frac{1}{2}\frac{dv}{dt} = 8\cos t - 4\sin t$$

$$\frac{d^2v}{dt^2} + 7\frac{dv}{dt} + 10v = 80\cos t - 40\sin t$$

(b) (**8 pts**) Determine $v(t)$ for all times.

solution:

For the differential equation, we have the following characteristic equation

$$s^2 + 7s + 10 = 0, \Rightarrow s_1 = -2,\ s_2 = -5$$

We have

$$v(t) = v_c(t) + v_p(t),$$

where

$$v_c(t) = K_1 e^{-2t} + K_2 e^{-5t}$$

and

$$v_p(t) = A\cos t + B\sin t$$

Substituting $v_p(t)$ in the differential equation yields

$$-A\cos t - B\sin t - 7A\sin t + 7B\cos t + 10A\cos t + 10B\sin t = (9A-7B)\cos t + (9B-7A)\sin t$$

$$= 80\cos t - 40\sin t$$

$$9A + 7B = 80, \quad 63A + 49B = 560$$

$$9B - 7A = -40, \quad 81B - 63A = -360$$

$$B = 20/13,$$

$$A = 100/13$$

Overall,

$$v(t) = K_1 e^{-2t} + K_2 e^{-5t} + \frac{100}{13}\cos t + \frac{20}{13}\sin t$$

We also note the following initial conditions

$$v(0^+) = 0, \quad i(0^+) = 0$$

Noting $i = 2\cos t - \frac{1}{20}\frac{dv}{dt}$, we have

$$\frac{dv}{dt}(0^+) = 40\,\text{Volts/second}$$

Using these initial conditions, we obtain

$$K_1 + K_2 = -\frac{100}{13}, \quad 2K_1 + 2K_2 = -\frac{200}{26}$$

$$-2K_1 - 5K_2 = 40 - \frac{20}{13}$$

Manipulation yields

$$-3K_2 = 40 - \frac{240}{26}, \quad K_2 = -\frac{400}{39}, \quad K_1 = \frac{100}{39}$$

Finally, we have

$$v(t) = \frac{100}{39}e^{-2t} - \frac{400}{39}e^{-5t} + \frac{100}{13}\cos t + \frac{20}{13}\sin t,\ V$$

(c) (**2 pts**) Using the result of previous part or otherwise, determine $i(t)$ for all times.

solution:

$$i = 2\cos t - \frac{1}{20}\frac{dv}{dt}$$