Question No 2:

a →

$$z(t) = \int_T x(\tau)\, y(t-\tau)\, d\tau \quad (1)$$

applying analysis Eq to (1)

$$c_K = \frac{1}{T}\int_T z(t)\, e^{-j\frac{2\pi}{T}t}\, dt$$

$$c_K = \frac{1}{T}\int_T\int_T x(\tau)\, y(t-\tau)\, e^{-j\frac{2\pi}{T}t}\, d\tau\, dt$$

we can take $x(\tau)$ out

$$c_K = \frac{\cancel{T}}{\cancel{T}}\int_T x(\tau)\, \frac{1}{T}\int_T y(t-\tau)\, e^{-j\frac{2\pi}{T}t}\, d\tau\, dt$$

this is the fourier series coefficients of shifted $y(t)$ so can be replaced with $b_K e^{-j\frac{2\pi}{T}\tau}$

$$c_K = \frac{T}{T}\int_T x(\tau)\, b_K\, e^{-j\frac{2\pi}{T}\tau}\, d\tau$$

from analysis Eq

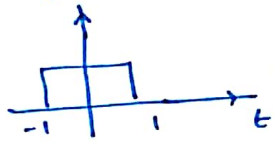
$$c_K = T\, b_K\, a_K$$

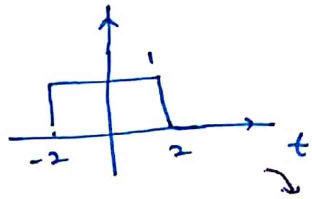
hence proved.

$\xrightarrow{b}$ By convolution integral we can confirm $z(t)$ can be obtained by the convolution of

$\Pi(t)$ and $\Pi(\frac{t}{2})$

rect function





for one period.

$$z(t) = \Pi(t) * \Pi(\frac{t}{2}) \rightarrow \text{for one period.}$$

So, from example 3.5 of book we can write

$$\Pi(t) \xrightarrow{FS} \frac{\sin(k\omega_0)}{k\pi} \qquad \omega_0 = \frac{2\pi}{6}$$

$$\Pi(\frac{t}{2}) \xrightarrow{FS} \frac{\sin(2k\omega_0)}{k\pi}$$

and by periodic convolution property just proved above

$$C_k = T \frac{\sin(k\omega_0)}{k\pi} \cdot \frac{\sin(2k\omega_0)}{k\pi}$$

$$C_k = \frac{6}{(k\pi)^2} \sin(k\frac{\pi}{3}) \sin(2k\frac{\pi}{3})$$

K

Question No 3

a) let

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jw_0 k t}$$

$$y(t) = \sum_{m=-\infty}^{\infty} b_m e^{jw_0 m t}$$

and

$$z(t) = \sum_{l=-\infty}^{\infty} c_l e^{jw_0 l t}$$

and we are given

$$z(t) = x(t)\, y(t)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k b_m e^{jw_0 (k+m) t}$$

let $k+m = l$ as $m \to \pm\infty$, $l \to \pm\infty$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k b_{l-k} e^{jw_0 l t}$$

rearranging terms

$$= \sum_{l=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_k b_{l-k} \right) e^{jw_0 l t}$$

↳ this is convolution of $c_l = a_k * b_k$

So,

$$z(t) = \sum_{l=-\infty}^{\infty} (a_K * b_K)\, e^{jw_0 l t}$$

by analysis Eq or inverse fourier series

$$c_l = a_K * b_K$$

or

$$c_l = \sum_{K=-\infty}^{\infty} a_K b_{l-K}$$

b)

we know

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jw_0 K t}$$

and by conjugation property

$$x(t)^* = \sum_{K=-\infty}^{\infty} a_K^* e^{jw_0 (-K) t}$$

So,

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)\, x^*(t)\, dt$$

$$= \frac{1}{T}\int_T x(t) \sum_{K=-\infty}^{\infty} a_K^* e^{-jw_0 K t} dt$$

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$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} a_k^* \int_T x(t) e^{-j\omega_0 k t} \, dt$$

from fourier series we know

$$\frac{1}{T}\int_T x(t) e^{-j\omega_0 k t} = a_k$$

$$= \sum_{k=-\infty}^{\infty} a_k^* \, a_k$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

hence,

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$\xrightarrow{c}$

$$h(t) = \begin{cases} e^{-2t} \cos(20\pi t) & -1 \le t \le 1 \\ 0 & 1 < t < 3 \end{cases}$$

h(t) can be thought of as

$h(t) = h(t)' * \cos(20\pi t)$ ⟶ also periodic with $T = 4$

multiplication

and $h(t)' = \begin{cases} e^{-2t} & -1 \le t \le 1 \\ 0 & 1 < t < 3 \end{cases}$

↳ period with $T = 4$

So, fourier series coefficients for $h(t)'$ will be

$$a_k' = \frac{1}{4}\int_{-1}^{1} e^{-(2+jw_0 t)}\, dt \quad \text{with } T_0 = 4$$

$$= \frac{1}{4}\left(\frac{-1}{2+jw_0}\right)\left(e^{-(2+jw_0)} - e^{(2+jw_0)}\right)$$

$$a_k' = \frac{-1}{4(2+jw_0)}\left(e^{-(2+jw_0)} - e^{(2+jw_0)}\right)$$

and b_K for $\cos(20\pi t)$ with $T = 4$

will be

$$\cos(20\pi t) = \frac{e^{40jw_0 t} + e^{-j40w_0 t}}{2}$$

$$a_{40} = 1/2$$
$$a_{-40} = 1/2$$

and a_k for $x(t)$ can be found by multiplication property as,

$$a_k = a_k' * \left(\frac{1}{2}\left(\delta(k-40) + \delta(k+40)\right)\right)$$

↳ Convolution

$$a_k = \frac{1}{2} a'_{k-40} + \frac{1}{2} a'_{k+40}$$

and a_k is FS for $h'(t)$

Question No 4

$\xrightarrow{a}$ $a_k = \begin{cases} 0 & k = 0 \\ (j)^k \dfrac{\sin(k\pi/4)}{k\pi} & k \neq 0 \end{cases}$

we know

$$j = e^{j\frac{\pi}{2}} \Rightarrow (j)^k = e^{j\frac{\pi}{2}k}$$

So,

$$a_k = \begin{cases} 0 & k = 0 \\ e^{j\frac{\pi}{2}k} \dfrac{\sin\left(k\frac{\pi}{4}\right)}{k\pi} & k \neq 0 \end{cases}$$

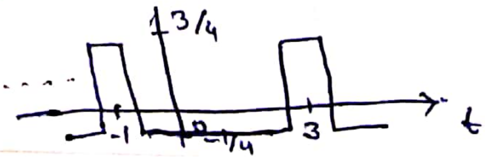
this looks like the FS of shifted square wave if $a_0 = \frac{1}{4}$ So we can write a_k as

$$a_k = \begin{cases} \frac{1}{4} - \frac{1}{4} & k = 0 \\ e^{j\frac{\pi}{2}k} \dfrac{\sin\left(k\frac{\pi}{4}\right)}{k\pi} & k \neq 0 \end{cases}$$

and

$$a_k \xrightarrow{FS} \Pi(2(t-2)) - \frac{1}{4}$$ for one period.

rect function

$x(t) = \Pi(2(t+2)) - \frac{1}{4}$ → plot 

$\underrightarrow{b}$:

$$a_k = (-1)^k \frac{\sin(k\pi/8)}{2k\pi} \quad , \quad a_0 = \frac{1}{16}$$

we know

$$(-1) = j^2 \quad \text{and} \quad j = e^{j\frac{\pi}{2}}$$

thus

$$j^2 = e^{j\pi} = -1$$

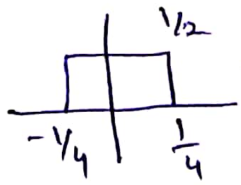
thus

$$a_k = e^{j\pi k} \frac{\sin(k\pi/8)}{2k\pi}$$

So, from example 3.5: we can say a_k are coefficient of shifted square wave

we know

$$\frac{1}{2} \alpha \frac{\sin\left(\frac{k\pi}{8}\right)}{k\pi} \xrightarrow{FS}$$



→ for one period

$$\frac{1}{2}\Pi(4t)$$

→ rect function

So by shifting property

$$a_k \xrightarrow{FS} \frac{1}{2}\Pi(4(t+2))$$

$$a_k = \begin{cases} 1 & k = \text{even} \\ 2 & k = \text{odd} \end{cases}$$

or

$$a_k = \begin{cases} 1 & \forall\ k \\ 1 & k = \text{odd} \end{cases}$$

for $a_k = 1 \quad \forall\ k$ means

$$a_k \xrightarrow{FS} \sum_{k=-\infty}^{\infty} \delta(t - 4k)$$ as signal is periodic with $T = 4$

and for

$a_k = 1$ for $k = \text{odd}$

or mathematically

$$a_k = \frac{a_k + (-1)^{k+1} a_k}{2} = \frac{a_k + e^{j\pi(k+1)} a_k}{2}$$

So, for $k = \text{odd}$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2}\delta(t-4k) + \frac{e^{j\pi}}{2} \sum_{k=-\infty}^{\infty} \delta(t+2-4k)$$

thus by linearity

$$a_k \xrightarrow{FS} \sum_{k=-\infty}^{\infty} \left(\frac{3}{2}\delta(t-4k) + \frac{e^{j\pi}}{2}\delta(t+2-4k) \right)$$

Question No 5

a)

$$x[n] = \sin\left[\frac{2\pi}{3}n\right]\cos\left[\frac{\pi}{2}n\right]$$

$$x[n] = \frac{1}{2}\sin\left(\frac{7\pi n}{6}\right) + \frac{1}{2}\sin\left(\frac{\pi n}{6}\right)$$

for

$$\sin\left[\frac{\pi}{6}n\right] \xrightarrow{FS} a_1 = \frac{1}{2j} \quad \text{with } T_0 = 12$$

$$\frac{e^{j\frac{2\pi}{12}n} - e^{-j\frac{2\pi}{12}n}}{2j} \qquad a_{-1} = -\frac{1}{2j}$$

Similarly for

$$\sin\left[7\frac{\pi}{6}n\right] \xrightarrow{FS} a_7 = \frac{1}{2j} \quad \text{with } T_0 = 12$$

$$a_{-7} = -\frac{1}{2j}$$

So for $x[n]$

$$x[n] \xrightarrow{FS} a_k = \begin{cases} \frac{1}{4j} & k = 1, 7 \\ -\frac{1}{4j} & k = -1, -7 \\ 0 & \text{else} \end{cases}$$

b →

$$x[n] = 1 - \sin\left[\frac{\pi}{4} n\right]$$

$$a_k = \delta(k)$$

and for $\sin\left[\frac{\pi}{4} n\right]$

$$a_k = \frac{1}{N} \sum_{N} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{12} \cdot \frac{1}{2j} \sum_{n=0}^{11} \left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right) e^{-jk\frac{\pi}{6}n}$$

$$= \frac{1}{24j} \sum_{n=0}^{4} \left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right) e^{-jk\frac{\pi}{6}n}$$

by simplification and using super position we can get a_k for $x[n]$

Question no 6

$$x[n] \xrightarrow{FS} a_K$$

a) by linearity and time reversal property

$$A\,x[n] + B\,x[-n] \xrightarrow{FS} A a_K + B a_{-K}$$

b) by frequency shifting property

$$x[n]\, e^{jL\left(\frac{2\pi}{N}\right)n} \xrightarrow{FS} a_{K-L}$$

c) shift in time domain

$$x[n-n_0] \xrightarrow{FS} a_K e^{-jK\left(\frac{2\pi}{N}\right)n_0}$$

d) by linearity and time shifting

$$x[n] + x\left[n-\frac{N}{2}\right] \xrightarrow{FS} a_K + a_K e^{-j\pi}$$

e) by conjugation property

$$\left(x[-n]\right)^* \xrightarrow{FS} a_K^*$$

f →

$$(-1)^n \quad x[n]$$

$$-1 = j^2 \Rightarrow e^{j\pi}$$

thus

$$(-1)^n x[n] = e^{j\pi n} x[n]$$

by frequency shifting property

$$(-1)^n x[n] \xrightarrow{FS} a_{k-\frac{N}{2}}$$

g →

$$y[n] = \begin{cases} x[n] & n = \text{even} \\ 0 & n = \text{odd} \end{cases}$$

or

$$y[n] = \frac{x[n] + (-1)^n x[n]}{2} = \frac{x[n] + e^{j\pi n} x[n]}{2}$$

So,

frequency shifting property.

assuming N is even.

$$a_k = \frac{a_k}{2} + \frac{1}{2}\left(a_{k-\frac{N}{2}}\right)$$

Problem # 1

a)

$$x(t) = \begin{cases} \frac{t}{2} & 0 \le t \le 2 \\ 1 & 3 \le t \le 2 \\ t-4 & 3 \le t \le 4 \end{cases}$$

for one period $[0 \quad 4]$

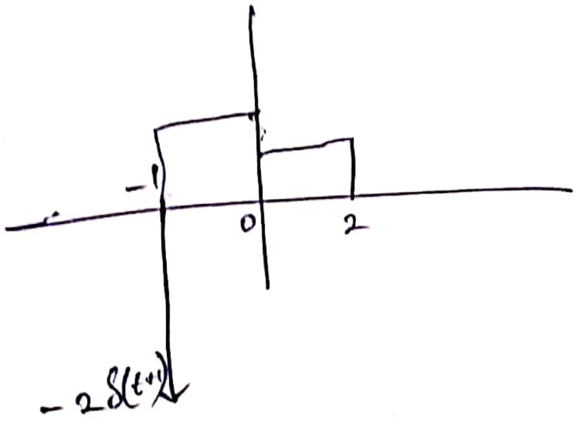
$$a_k = \frac{1}{4} \int_0^4 x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{4} \left[\int_0^2 \frac{t}{2} e^{-jk\omega_0 t}\, dt + \int_2^3 e^{-jk\omega_0 t}\, dt + \int_3^4 (t-4) e^{-jk\omega_0 t}\, dt \right]$$

$$a_k = \frac{1}{4} \left[\left(\frac{j\omega_0 k t + 1}{2} \right) e^{-jk\omega_0 t} \Big|_0^2 + \frac{-1}{j\omega_0 k} e^{-jk\omega_0 t} \Big|_2^3 \right.$$
$$\left. + (j\omega_0 k t + 1) e^{-jk\omega_0 t} \Big|_3^4 + \frac{-4}{jk\omega_0} e^{-jk\omega_0 t} \Big|_3^4 \right]$$

By simplification one can find same answer as found by properties.

1-b →
derivative of $x(t)$ for one period

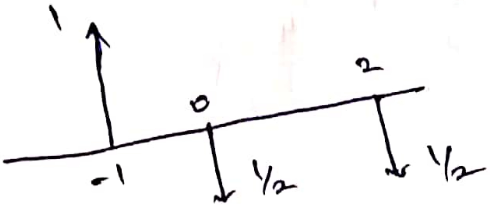


$\frac{d}{dt} x(t)$

now,

$$\frac{d}{dt} x(t) = -2\delta(t+1) + \underbrace{\Pi(t-1)}_{\text{rect}} + \Pi\left(2t-\tfrac{1}{2}\right)$$

now taking derivative of rect function



$\frac{d^2}{dt^2} x(t)$ ↓ without previous impulse.

So, FS for $\frac{dx}{dt}$ will be

$$a_k' = \frac{1}{4}\left(-\frac{1}{2}e^{jk\omega_0} + e^{jk\omega_0} - \frac{1}{2} - \frac{1}{2} e^{-j2k\omega_0} \right) \alpha \frac{1}{jk\omega_0}$$

and farrier series coefficient for $x(t)$ can be found by dividing ~~at~~ a_k again by

$jk\omega_0$.

So,

$$a_k = \frac{-\frac{1}{2}\cancel{e^{jk\omega_0}}}{\cancel{jk\omega_0}} + \frac{1}{4}\left(e^{jk\omega_0} - \frac{1}{2} - \frac{1}{2}e^{-2jk\omega_0}\right)\alpha\frac{1}{(jk\omega_0)^2}$$

$$a_k = -\frac{1}{2} + \frac{1}{4}\left(\frac{e^{jk\omega_0} - \frac{1}{2} - \frac{1}{2}e^{-2jk\omega_0}}{(jk\omega_0)^2}\right)$$