Discrete-time Fourier Transform (DTFT)

Recap :-

Time	Frequency
- Continuous Periodic	Discrete Aperiodic (FS)
	Impulse (Discrete) Aperiodic (FT)
- Discrete Periodic	Discrete Periodic (FS)
- Continuous Aperiodic	Continuous Aperiodic (FT)

- Now, we want to represent DT Aperiodic signal in terms of complex exponentials.
- What do you expect about frequency domain description ?

Discrete Aperiodic (time) $\longleftrightarrow$? Continuous Periodic (frequency)

- Define Analysis / Synthesis Equation
- Analysis Equation (DT Fourier Transform)

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

- Synthesis Equation

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

- Note the integral in synthesis equation is over 2π which is due to the fact that $X(e^{j\omega})$ is periodic with period 2π.

- It can be easily shown that $X(e^{j\omega})$ is periodic with period 2π using analysis equation

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = X(e^{j\omega}) \quad \underline{\text{QED}}$$

- It is also worth noting that $\omega = 0 = 2\pi$ for DT signal. Consequently, highest frequency in DT signal is $\omega = \pi$. $e^{j\pi n}$ has the maximum variation in DT.

- You can review textbook for the derivation of analysis/synthesis equation. I think the developments made here are very intuitive and makes sense.

- let's determine DTFT of some well-known signals

Example 1

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = 1$$

Example 2 $\quad x[n] = \delta[n-n_0]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-n_0]\, e^{-j\omega n} = e^{-j\omega n_0}$$

$$= \cos(\omega n_0) - j\sin(\omega n_0)$$

Periodic with period 2π.

Example 3

$$x[n] = \alpha^n u[n] \qquad |\alpha|<1$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n]\, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-jw}\right)^n \qquad \text{(Infinite Geometric Series)}$$

$$= \frac{1}{1-\alpha e^{-jw}}$$

$\downarrow$

Periodic (obviously!)

Example 4

- If $x[n] = \alpha^n u[n] \qquad |\alpha|<1$

Determine the Fourier transform of y[n] given by

$$y[n] = x[n] + \alpha x[-n-1]$$

(You can reuse Example 3 the result and properties)

Since we have not studied properties yet, we find $Y(e^{jw})$ by definition

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-jwn} + \sum_{n=-\infty}^{-1} \alpha^{-n} e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-jw}\right)^n + \sum_{k=1}^{\infty} \left(\alpha e^{jw}\right)^k \qquad \left[\text{Replace } n=-k\right]$$

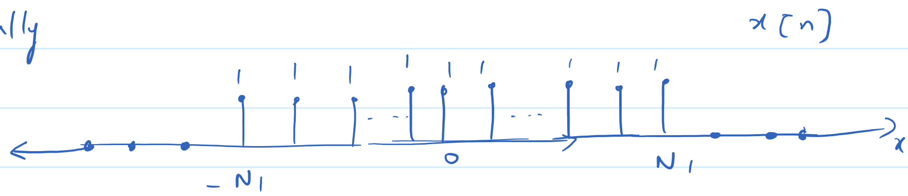
$$= \frac{1}{1-\alpha e^{-jw}} + \frac{\alpha e^{jw}}{1-\alpha e^{jw}}$$

$$= \frac{1-\alpha^2}{1-2\alpha\cos w+\alpha^2}$$

Example 5 (Square Pulse)

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$$

Graphically



$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} = \sum_{n=-N_1}^{N_1} e^{-jwn}$$

$$= \frac{e^{jwN_1}\left(1 - e^{-jw(2N_1+1)}\right)}{1-e^{-jw}}$$

$$= \frac{e^{jw(N_1+1/2)} - e^{-jw(N_1+1/2)}}{e^{jw/2} - e^{-jw/2}}$$

$$= \frac{2j\,\sin\left(w\left(N_1+\frac{1}{2}\right)\right)}{2j\,\sin(w/2)}$$

$$= \frac{\sin\left(w\left(N_1+\frac{1}{2}\right)\right)}{\sin(w/2)}$$

- Plots shown in Lecture.
- $X(e^{jw})$ is the sum of infinite shifted sincs.

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- You should reflect on the statement above.

- CONVERGENCE OF DT FT

Since we have DT signal, we cannot run into issues like infinite number of maxima/minima and discontinuities. We have two conditions of convergence

1. $\sum_{n=-\infty}^{\infty} |x[n]| < \infty \implies X(e^{jw}) \rightarrow$ valid.

(Absolute Summable)

2. $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \implies X(e^{jw})$ — valid

(Absolute Square-Summable)

PROPERTIES OF DT FT.

Notation;
$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$
$$y[n] \xleftrightarrow{F} Y(e^{j\omega})$$

- Linearity;

$$\alpha x[n] + \beta y[n] \longleftrightarrow \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

- Time-shift

$$x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})$$

Proof: use analysis equation.

- Frequency-shift

$$x[n]\, e^{-j\omega_0 n} \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

Proof; use synthesis equation

Example 5.7; establishes relationship b/w impulse responses of ideal low-pass and high-pass filters.

- Conjugation

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

Proof; QED, using synthesis equation

- Difference

$$x[n] - x[n-1] \longleftrightarrow \left(1 - e^{-j\omega}\right) X(e^{j\omega})$$

Proof; use time-shift and linearity properties

- Accumulation (Anti-difference)

$$\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Proof: (Informal)

- $\dfrac{X(e^{jw})}{1-e^{-jw}}$ due to anti-difference nature of accumulation.

- $\pi X(e^{j0})$ due to the fact that DC term gets lost in derivative or difference

- Summation; due to periodicity of FT.

Example; FT of $u[n]$

$$x[n] = u[n] = \sum_{k=-\infty}^{n} z[k] \quad , \quad z[n] = \delta[n]$$

$$\Rightarrow \quad Z(e^{jw}) = 1$$

$$x(e^{jw}) = \frac{Z(e^{jw})}{1-e^{-jw}} + \pi Z(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(w-2\pi k)$$

$$X(e^{jw}) = \frac{1}{1-e^{-jw}} + \pi \sum_{k=-\infty}^{\infty} \delta(w-2\pi k)$$

- Time Expansion:-

- We have DT signals; compression or expansion is not defined analogous to CT signals.

- Define time expansion by a factor of 'k' as inserting $(k-1)$ zeros between signal samples

- Mathematically

$$x_{(k)}[n] = \begin{cases} x\left[\frac{n}{k}\right] & n = \ell k \quad \ell \in Z \\ 0 & \text{otherwise.} \end{cases}$$

$$x_{(k)}[n] = \begin{cases} x\left[\frac{n}{k}\right] & n = lk \quad l \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

- FT of $x_{(k)}[n]$?

Let $x_{(k)}[n] \xleftrightarrow{F} X_{(k)}(e^{jw})$

Using analysis equation:

$$X_{(k)}(e^{jw}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-jwn}$$

Let $n = pk$

$$X_{(k)}(e^{jw}) = \sum_{p=-\infty}^{\infty} x_{(k)}[pk] e^{-jwpk}$$

$$= \sum_{p=-\infty}^{\infty} x[p] e^{-jwpk}$$

$$= X(e^{jwk})$$

- FT compresses by a factor of k.
- $X(e^{jwk})$ — periodic with period $\frac{2\pi}{k}$
- Expansion in time domain $\longleftrightarrow$ Compression in frequency domain

- Differentiation in frequency

$$-jn \; x[n] \xleftrightarrow{F} \frac{d}{dw} X(e^{jw})$$

Proof: Use analysis equation

- Parseval's Relation

- Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

- Example 5.9, 5.10

- CONVOLUTION PROPERTY

$$x[n] * y[n] \xleftrightarrow{F} X(e^{j\omega})\, Y(e^{j\omega})$$

Proof: Similar to CT FT convolution property

CONSEQUENCES:

- As mentioned earlier, convolution property enables simpler interpretation of convolution in frequency domain.

- Convolution; twisted, complicated in time domain
  untwisted, simpler in frequency domain

- Since convolution is just multiplication in frequency domain, action of LTI systems as filters is very easy to interpret.

- Filters _ LTI systems which act as window in frequency domain.

- Examples 5.11, 5.12, 5.13 (covered in lectures)

- MULTIPLICATION PROPERTY

$$x[n]\, y[n] \xleftrightarrow{F} \frac{1}{2\pi}\, X(e^{j\omega}) * Y(e^{j\omega})$$

Periodic convolution

$$= \frac{1}{2\pi} \int_{2\pi} X(e^{j\eta})\, Y\left(e^{j(\omega-\eta)}\right) d\eta$$

DTFT Properties (contd.), Examples, Analysis of LTI Systems

- We looked at properties of DTFT in the last lecture.

- we only need to cover one more property: Time expansion.

- Since we have a DT signal, we can only define expansion for an integer k as

$$x_{(k)}[n] = \begin{cases} x[n/k] & n = \dots -2k, -k, 0, k, 2k \dots \\ 0 & \end{cases}$$

- Expanded signal by a factor of k.
- Simply insert $(k-1)$ zeros b/w samples of $x[n]$ to get $x_{(k)}[n]$.

- Let $x_{(k)}[n] \xleftrightarrow{DTFT} X_{(k)}(e^{j\omega})$

- By definition;

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n}$$

$$= \sum_{p=-\infty}^{\infty} x_{(k)}[pk] e^{-j\omega pk}$$

$$= \sum_{p=-\infty}^{\infty} x[p] e^{-j(\omega k)p}$$

$$X_K(e^{j\omega}) = X(e^{j\omega k})$$

- Compressed in frequency by a factor k.
- Expansion in DT $\longleftrightarrow$ Compression in frequency.

- **DIFFERENTIATION IN FREQUENCY PROPERTY**

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$-jn\, x[n] \longleftrightarrow \frac{d}{d\omega}\left(X(e^{j\omega})\right)$$

- Can be shown using analysis equation.

- Table 5.1 lists all of the properties, which you must know how to apply.

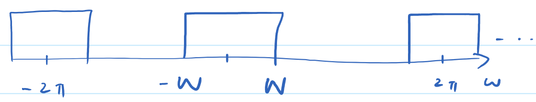
- Commonly used DTFT pairs are tabulated in Table 5.2. You should know how to derive these and importantly use these in conjunction with the properties of DTFT.

- In this lecture, we mostly focus on Examples.

<u>Example 01</u> Relation b/w impulse responses of DT low-pass and high pass filters.

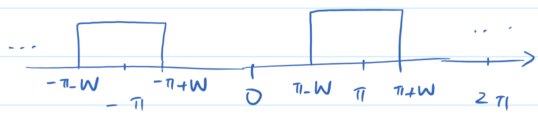
(Example 5.7, textbook)

<u>LOW PASS</u>: $H_{lp}(e^{jw})$ ...



W is the cut-off frequency.

HIGH-PASS. $H_{hp}(e^{jw})$



- Remember π is the highest frequency of the DT signal.

clearly
$$H_{lp}(e^{jw}) = H_{hp}\left(e^{j(w-\pi)}\right)$$

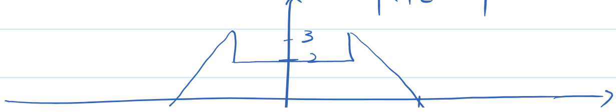
Using Frequency shift property
$$h_{lp}[n] = h_{hp}[n]\, e^{-j\pi n}$$
$$= (-1)^n\, h_{hp}[n]$$

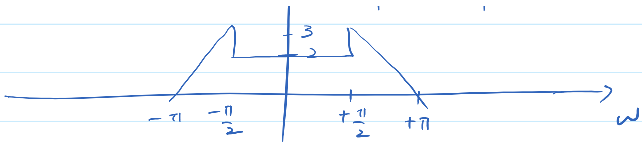
Interpretation; Do it yourself!

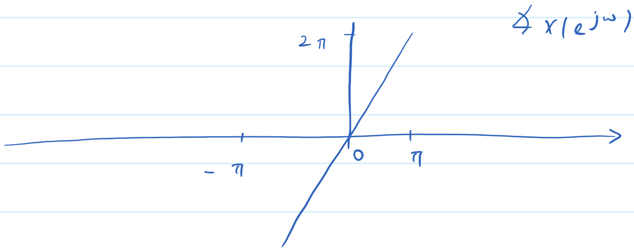
<u>Example 02</u> (5.10, textbook)
Given $X(e^{jw})$, determine if $x[n]$ is periodic, real, even, finite energy.

<u>$X(e^{jw})$ for $-\pi \le w \le \pi$:</u>

$|X(e^{jw})|$







Periodic ? Not periodic, because it is continuous in frequency. If we have impulses in frequency, we would have periodic signal in time.

Real ? Yes, Due to conjugate symmetry

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

$\Rightarrow$ Magnitude even

$\Rightarrow$ Phase odd

Even ; No, FT is not real/even

Finite Energy ? Yes, followed from Parseval's relation.

- DUALITY – Final (re)-visit

Time		Frequency		
CT	Periodic	Discrete	Aperiodic	(CT FS)
		Discrete (Impulses)	Aperiodic	(CT FT)
DT	Periodic	Discrete	Periodic	(DT FS)

DT	Periodic	Discrete	Periodic	(DT FS)
		Discrete (Impulses)	Periodic	(DT FT)
CT	Aperiodic	Continuous	Aperiodic	(CT FT)
DT	Aperiodic	Continuous	Periodic	(DT FT)

→ Duality between CTFT, time and frequency (Studied in detail)

→ Duality between DT FS, time and frequency

→ Duality between DT FT and CT FS

discrete, aperiodic in time
continuous/periodic in frequency

discrete/aperiodic in frequency
continuous/periodic in time.

- ANALYSIS OF LTI SYSTEMS USING DTFT.

  - DT LTI system can be equivalently described by a Linear Constant Coefficient (LCC) differential equation given by:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

- Here $x[n]$ and $y[n]$ denotes input and output of the LTI system respectively.

- DTFT or Z-transform (to be studied later) helps us in analyzing and finding impulse response of LTI system given in the form above.

- By applying DTFT on the LCC difference equation and using time-shift property:

$$\sum_{k=0}^{N} a_k e^{-jwk} Y(e^{jw}) = \sum_{k=0}^{M} b_k e^{-jwk} X(e^{jw})$$

$$H(e^{jw}) \triangleq \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{M} b_k e^{-jwk}}{\sum_{k=0}^{N} a_k e^{-jwk}}$$

- Following convolution property, $H(e^{jw})$ is the DTFT of impulse response $h[n]$ of the system

- Given LCC difference equation, $h[n]$ can be determined by first finding $H(e^{jw})$ followed by taking its inverse DTFT.

- Example 5.18, $\underbrace{5.19, 5.20}_{\text{Covered in lecture.}}$