Lahore University of Management Sciences
Department of Electrical Engineering, Syed Babar Ali School of Science and Engineering

EE310 - Signals and Systems

Spring 2019 – May 16, 2019

Final Exam

Time Allowed: 180 minutes **Total Points**: 100

Student ID ____________________________ Signature ________________________________

Instructions

1. Do not flip this page over until told to do so.
2. The exam is closed-book, closed-notes. Two hand-written A4-sized formula sheets (two-sided) are allowed. Calculators are also allowed.
3. Try to identify the easiest way to solve a problem.
4. Clearly outline all your steps. Solutions with inadequate justifications and/or steps may not receive full credit.
5. **The exam consists of TWO Parts.**
   (a) The first part is worth 30 pts, and is to be solved on the exam sheet. It is to be returned to the exam staff in the first 45 minutes.
   (b) The second part is worth 70 pts and is also to be solved on the exam sheet. Blue answer books are only provided for rough work. Do not return them after the exam.
6. **Note**: CT and DT refers to continuous-time and discrete-time, respectively.

Problem 1	[30]	
Problem 2	[20]	
Problem 3	[10]	
Problem 4	[20]	
Problem 5	[10]	
Problem 6	[10]	
Total	[100]	

EE310 – Signals and Systems
Final Examination Spring 2019

Part - 1 (30 pts)

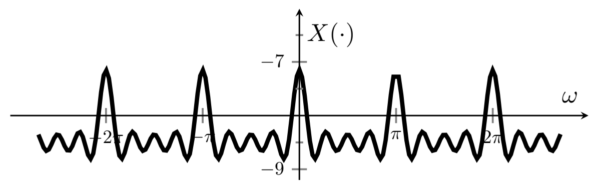
Student ID ____________________________ Name ________________________________

Section Instructor ____________________ Signature ________________________________

Instructions: This part needs to be solved on this sheet, and is to be returned to the exam staff in the first 45 minutes.

Problem 1. Select ALL (upto three) correct choices. Every incorrect answer would earn a penalty of 2/3 points but the total marks of any multiple choice question will not be less than zero.

(1) Based on the structure of the periodic frequency domain representation $X(\cdot)$ (period 2π) shown to the right, which of the following is the best description of the time domain signal?



(a) The time domain signal is discrete-time.
(b) The time domain signal is not periodic in time.
(c) The time domain signal is continuous-time.
(d) The time domain signal is periodic in time.

(2) The convolution

$$\frac{\sin 10t}{\pi t} * \frac{\sin 6t}{\pi t} * \frac{\sin 12t}{\pi t}$$

is equal to

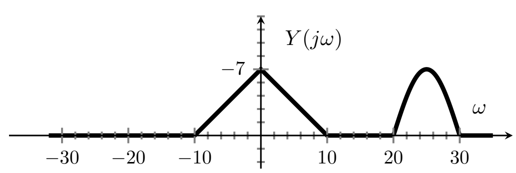
(a) $\frac{\sin 6t}{\pi t} * \frac{\sin 12t}{\pi t}$

(b) $\frac{\sin 12t}{\pi t}$

(c) $\frac{\sin 6t}{\pi t}$

(d) $\frac{\sin 10t}{\pi t} * \frac{\sin 6t}{\pi t}$

(3) The Fourier transform of the signal $y(t)$, denoted by $Y(j\omega)$, is shown to the right. $Y(j\omega)$ is purely real. Which of the following statement(s) about the signal $y(t)$ must be true?



(a) $y(t)$ is purely imaginary, that is, $y(t) = -y^*(t)$.
(b) $y(t)$ is complex conjugate symmetric, that is, $y(t) = y^*(-t)$.
(c) $y(t)$ is real valued, that is, $y(t) = y^*(t)$.
(d) $y(t)$ is complex valued.

(4) A discrete-time LTI system with input $x[n]$ and output $y[n]$ is given by

$$y[n] = x[n] + \alpha\, x[n-1] + \beta\, x[n-2]$$

for some constants α and β. What values of α and β yield a filter with frequency response at $\omega = 0$ of zero, and frequency response at $\omega = \pi$ of 1?

(a) $\alpha = 0$ and $\beta = -1$
(b) $\alpha = -1/2$ and $\beta = -1/2$
(c) $\alpha = 1$ and $\beta = 0$
(d) $\alpha = -1$ and $\beta = 0$

(5) If the Fourier transform of the signal $x(t) = u(t) - u(t-3)$ is denoted by $X(j\omega) = A(j\omega)e^{jB(j\omega)}$, $B(j\omega)$ is given by:

(a) $-\omega/2$
(b) $-3\omega/2$
(c) $3\omega/2$
(d) 3ω

(6) If $y[n] = \sum_{m=-\infty}^{n} x[m] \times \delta[-n_0 - m]$, then $y[n]$ equals

(a) $x[-n_0]u[n-n_0]$
(b) $x[-n_0]u[n+n_0]$
(c) $x[-n_0]$
(d) $x[-n_0]u[-n-n_0]$

(7) If the transfer function for a system is given as $H(s) = \frac{e^{2\pi s}}{s+5}$ with ROC $Re\{s\} > -5$, then the system

(a) is a non-causal system
(b) is an unstable system
(c) is a causal system
(d) is a stable system

(8) For a stable system with a *real* impulse response $h(t)$ and system function $H(s)$, which of the following statement can not be true.

(a) The system has exactly two poles at $s = -5 + j3$ and $s = -5 + j6$
(b) The system has exactly two poles at $s = -6$ and $s = +6$
(c) The system has exactly one pole at $s = -6$
(d) The system has exactly one pole at $s = +6$

(9) A linear time-invariant (LTI) system is given by the following convolution $y(t) = \text{sinc}(3t) * x(t)$, where $x(t)$ is the CT input signal, $y(t)$ is the CT output signal. The system is

(a) a low pass filter.
(b) a high pass filter.
(c) an all pass filter.
(d) a band-pass filter.

(10) For an *unstable* DT system with a impulse response $h[n]$ and system function $H(z)$, which of the following statement can not be true.

(a) The system has exactly one pole at $z = -2$ and is causal
(b) The system has exactly one pole at $z = +6$
(c) The system has exactly two poles at $z = -0.5$ and $z = -0.8$ and is causal
(d) The system has exactly two poles at $z = -0.5$ and $z = +0.5$

(11) If the Fourier transform of the signal $x[n] = \left(\frac{1}{2}\right)^n u[n]$ is denoted by $X(e^{j\omega})$, $\int_{2\pi} |X(e^{j\omega})|^2 d\omega$ is equal to:

(a) $\pi/3$

(b) $8\pi/3$

(c) $4/3$

(d) 4π

(12) A linear time-invariant (LTI) system is given by the following convolution

$$y(t) = e^{j\pi t/4} * x(t),$$

where $x(t)$ is the CT input signal, $y(t)$ is the CT output signal.
Which of the following is/are true?

(a) The system is a low pass filter.

(b) The system is a high pass filter.

(c) The system is a band-pass filter.

(d) None of the above.

(13) If the output of an LTI system is $e^{j\omega n} + e^{j\omega(n-1)}$ for input $e^{j\omega n}$ and any ω, the discrete-time Fourier transform of the impulse response of the system is

(a) $e^{j\omega}$

(b) $1 + e^{-j\omega}$

(c) $1 + e^{j\omega}$

(d) Cannot be determined using the information provided.

(14) The signal $f(t)$ has Fourier transform denoted by $g(\omega)$. The Fourier transform of $g(t)$ is

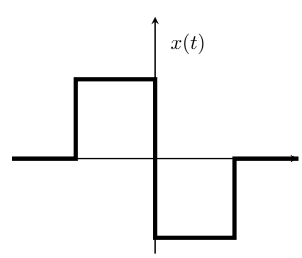
(a) $2\pi f(\omega)$

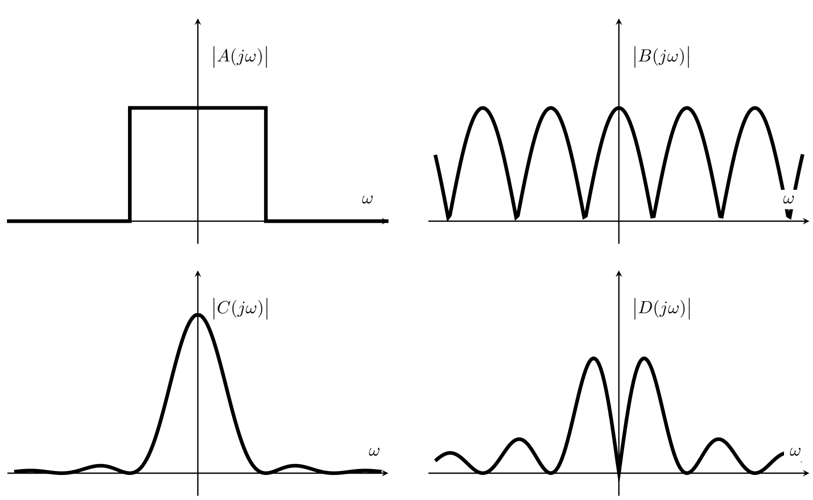
(b) $2\pi f(-\omega)$

(c) $\frac{1}{2\pi} f(\omega)$

(d) $f(-\omega)$

(15) For a signal $x(t)$ shown to the right, choose the magnitude of its Fourier transform from the four candidates $|A(j\omega)|$, $|B(j\omega)|$, $|C(j\omega)|$ and $|D(j\omega)|$ below. Circle the correct choice. (The figures are not to any particular scale and the question can be answered based on the general shapes given) D





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Part - 2 (70 pts)

Problem 2. (**20 pts**)

(a) (**5 pts**) Determine the Fourier transform $X(j\omega)$ of the signal $x(t)$ shown in the Figure below.

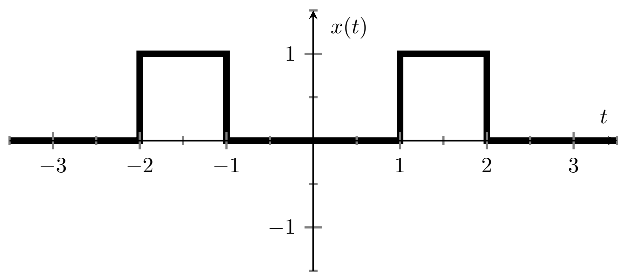


Figure 1: Plots of $X(j\omega)$.

We have

$$x(t) = g(t+1.5) + g(t-1.5),$$

where $g(t)$ is the rectangular pulse of width 1 and centered at 0. We have $G(j\omega) = 2\frac{\sin(\omega/2)}{\omega}$.

Using time-shift property

$$X(j\omega) = G(j\omega)\left(e^{j\omega 1.5} + e^{-j\omega 1.5}\right) = 4\frac{\sin(\omega/2)}{\omega}\cos(1.5\omega).$$

(b) (**8 pts**) Using the result of part (a) or otherwise, determine the Fourier transform $Y(j\omega)$ of the signal $y(t)$, where$y(t)$ is of the form $y(t) = a(t)e^{jb(t)}$, where $a(t)$ and $b(t)$ are shown in Figure 1.

(**Hint:** First determine the Fourier transform corresponding to the signal $a(t)$ and use the property of Fourier transform to find the Fourier transform of $y(t)$.)

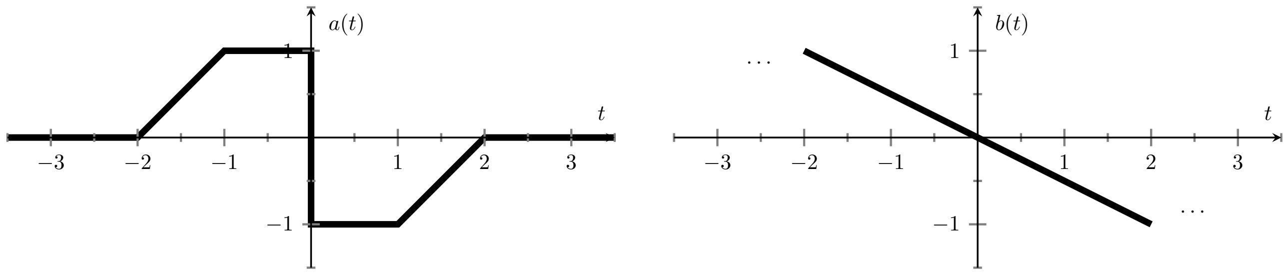


Figure 2: Plots of $A(j\omega)$ and $B(j\omega)$.

Assume $a'(t) = z(t)$. By inspection $z(t) = x(t) - 2\delta(t)$. We first compute

$$Z(j\omega) = X(j\omega) - 2.$$

Using integration property:

$$A(j\omega) = \frac{Z(j\omega)}{j\omega} + \pi Z(j0)\delta(\omega) = \frac{Z(j\omega)}{j\omega} = 4\frac{\sin(\omega/2)}{j\omega^2}\cos(1.5\omega) - \frac{2}{j\omega},$$

where we note that $Z(j0) = X(j0) - 2 = 2 - 2 = 0$.

Now we compute the Fourier transform of

$$y(t) = a(t)e^{jb(t)} = a(t)e^{-jt/2},$$

using frequency shift property as

$$Y(j\omega) = A(j(\omega + 0.5)).$$

(c) (**3 pts**) The following differential equation relates the output $y(t)$ of causal continuous LTI system to the input $x(t)$

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

Determine the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$.

By applying Fourier transform

$$j\omega Y(j\omega) + 3Y(j\omega) = X(j\omega)$$

which yields

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{3 + j\omega}.$$

(d) (**4 pts**) If the input to the LTI system given in part (c) is $x(t) = e^{-3t}u(t)$, determine $Y(j\omega)$ and $y(t)$.

We first determine $X(j\omega)$:

$$X(j\omega) = \frac{1}{3 + j\omega}$$

Now, determine $Y(j\omega)$ as follows:

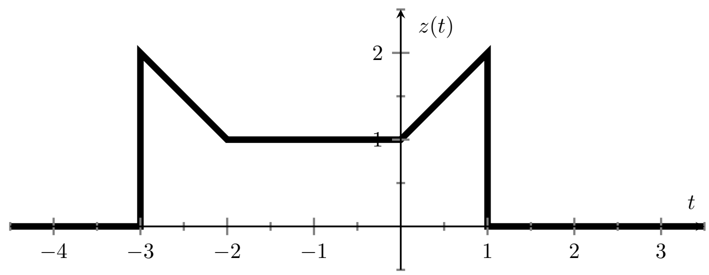
$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(3 + j\omega)^2}.$$

Taking inverse transform yields

$$y(t) = te^{-3t}u(t).$$

Problem 3. (**10 pts**) For a signal $z(t)$ shown below whose Fourier transform is denoted by $Z(j\omega)$, determine the following: (show your working)

Hint: You do not need to evaluate the Fourier transform $Z(j\omega)$ to answer this question. You can use the properties of the Fourier transform and synthesis/analysis equations.



(a) (**2 pts**) $\angle Z(j\omega)$ We first note that $z(t)$ is an even signal, say $y(t)$, shifted to the left (time-advanced) by 1, that is, $z(t) = y(t+1)$. Using the time-shift property, we have $Z(j\omega) = Y(j\omega)e^{j\omega}$ or $\angle Z(j\omega) = \angle Y(j\omega) + \omega = \omega$, noting $y(t)$ is an even signal which implies $\angle Y(j\omega) = 0$. Thus, $\angle Z(j\omega) = \omega$.

(b) (**3 pts**) $\displaystyle\int_{-\infty}^{\infty} Z(j\omega)e^{j\omega/2}\,d\omega$

For $Y(j\omega) = Z(j\omega)e^{j\omega/2}$, we have $y(t) = z(t+0.5)$. $\displaystyle\int_{-\infty}^{\infty} Z(j\omega)e^{j\omega/2}\,d\omega = 2\pi y(0) = 2\pi z(0.5) = 3\pi$.

(c) (**5 pts**) $\displaystyle\int_{-\infty}^{\infty} Z(j\omega)\,e^{j\omega}\,\frac{2\sin(2\omega)}{\omega}\,d\omega$

Let $Y(j\omega) = Z(j\omega)e^{j\omega}$ and $X(j\omega) = \frac{2\sin(\omega 2)}{\omega}$. We have $y(t) = z(t+1)$ and $x(t)$ given by

$$x(t) = \begin{cases} 1, & |t| < 2 \\ 0, & |t| > 2 \end{cases}.$$

$$\int_{-\infty}^{\infty} Z(j\omega)\,e^{j\omega}\,\frac{2\sin\omega}{\omega}\,d\omega = 2\pi x(t) * y(t)\big|_{t=0} = 2\pi \int x(\tau)y(-\tau)d\tau = 5\pi.$$

Problem 4.

(a) (**8 pts**) Suppose we wish to design a discrete-time LTI system which has the property that if the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1],$$

then the output is

$$y[n] = \left(\frac{1}{3}\right)^n u[n].$$

Find the impulse response $h[n]$ of the LTI system that satisfies the property mentioned above.

We have

$$\begin{aligned} X\left(e^{jw}\right) &= \frac{1}{1-0.5e^{-jw}} - \frac{e^{-jw}}{4(1-0.5e^{-jw})} \\ &= \frac{1-0.25e^{-jw}}{1-0.5e^{-jw}} \end{aligned}$$

Also,

$$Y\left(e^{jw}\right) = \frac{1}{1-\frac{e^{-jw}}{3}}$$

Therefore,

$$H\left(e^{jw}\right) = \frac{Y\left(e^{jw}\right)}{X\left(e^{jw}\right)} = \frac{1-0.5e^{-jw}}{(1-\frac{e^{-jw}}{3})(1-0.25e^{-jw})}$$

Using partial fraction expansion, we have

$$H\left(e^{jw}\right) = \frac{-2}{1-\frac{e^{-jw}}{3}} + \frac{3}{1-0.25e^{-jw}}$$

As a result, the impulse response of the system is given by

$$h[n] = -2\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{4}\right)^n u[n]$$

(b) (**5 pts**) Consider a discrete-time LTI system with a real impulse response $h[n]$ and a frequency response $H\left(e^{jw}\right)$. If the input to the LTI system is

$$x[n] = \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{4}n\right),$$

the output is observed to be

$$y[n] = 0.5\cos\left(\frac{\pi}{2}\left(n+\frac{\pi}{4}\right)\right).$$

Do we have enough information to determine $H\left(e^{j\pi/2}\right)$ (both phase and magnitude)? If your answer is yes, determine (with proper justification) $H\left(e^{j\pi/2}\right)$. If your answer is no, provide clear reasoning.

Yes, we have enough information to determine $H\left(e^{j\pi/2}\right)$. If $x[n] = \cos(w_0 n)$ is the input to an LTI system, we have

$$\begin{aligned}
y[n] &= \frac{1}{2}e^{jw_0 n} \star h[n] + \frac{1}{2}e^{-jw_0 n} \star h[n] \\
&= \frac{1}{2}e^{jw_0 n} H\left(e^{jw_0}\right) + \frac{1}{2}e^{-jw_0 n} H\left(e^{-jw_0}\right) \\
&= \frac{1}{2}\left|H\left(e^{jw_0}\right)\right| e^{j(w_0 n + \phi(w_0))} + \left|H\left(e^{jw_0}\right)\right| e^{-j(w_0 n + \phi(w_0))} \\
&= \left|H\left(e^{jw_0}\right)\right| \cos\left(w_0 n + \phi(w_0)\right)
\end{aligned}$$

where $\phi(w)$ is the phase of the frequency response as a function of the frequency. The second equality is because of complex exponentials being Eigen functions of LTI systems, and the second equality because of the fact that $h[n]$ is real, and therefore the frequency response is conjugate symmetric. As result, we can easily see that

$$H\left(e^{j\pi/2}\right) = 0.5e^{j\pi^2/8}$$

(c) (**7 pts**) Let $x[n]$ be a discrete-time signal with Fourier transform $X\left(e^{jw}\right)$. Let

$$y[n] = \begin{cases} 3\,x\left[\frac{n}{2}\right] & n \text{ is even} \\ \\ 2\,x\left[\frac{n-1}{2}\right] & n \text{ is odd} \end{cases}$$

Determine $Y\left(e^{jw}\right)$ as a function of $X\left(e^{jw}\right)$.

We know that

$$x_{(2)}[n] = \begin{cases} x\left[\frac{n}{2}\right] & n \text{ is even} \\ \\ 0 & n \text{ is odd} \end{cases}$$

Shifting the signal, we get

$$x_{(2)}[n-1] = \begin{cases} x\left[\frac{n-1}{2}\right] & n-1 \text{ is even} \\ \\ 0 & n-1 \text{ is odd} \end{cases}$$

$$\Rightarrow x_{(2)}[n-1] = \begin{cases} x\left[\frac{n-1}{2}\right] & n \text{ is odd} \\ \\ 0 & n \text{ is even} \end{cases}$$

From this we conclude that $y[n] = 3x_{(2)}[n] + 2x_{(2)}[n-1]$, and therefore

$$Y\left(e^{jw}\right) = \left(3 + 2e^{-jw}\right) X\left(e^{j2w}\right)$$

Problem 5.

(a) (**4 pts**) Consider a continuous-time *causal* LTI system with an impulse response $h(t)$. For the unknown impulse response, we are told that the signal $h(t)e^{3t}$ is absolutely integrable. Given this information, Prof. Langdon claims that the system must be stable. Do you agree or disagree with the professor? Provide clear reasoning to support your answer (*No credit without correct justification*).

Yes, I agree with the professor. If $h(t)e^{3t}$ is absolutely integrable, then we must have

$$\left|\int_{-\infty}^{\infty} h(t)e^{3t}e^{-jwt}dt\right| = \left|\int_{-\infty}^{\infty} h(t)e^{-(-3+jw)t}dt\right| < \infty$$

As a result, we must have $s = -3 + jw$ inside the ROC for all w, i.e., we must have the $s = -3$ line inside the ROC. Since the system is causal, we must have all points to the right of the $s = -3$ line inside the ROC as well. Therefore, the jw-axis is inside the ROC, because of which the system must be stable.

(b) (**2 pts**) Consider an LTI system whose input-output relationship is governed by the differential equation

$$\frac{d^2y(t)}{dt} + \alpha\frac{dy(t)}{dt} + \alpha^2 y(t) = 2\frac{dx(t)}{dt} - x(t),$$

where α is some real number. Use the derivative property of the Laplace transform to determine the system's transfer function $H(s) = \frac{Y(s)}{X(s)}$.

Computing the Laplace transform on both sides, we get the system transfer function s

$$H(s) = \frac{2s-1}{s^2 + \alpha s + \alpha^2}$$

(c) (**4 pts**) If the system in part (b) is known to be causal, what condition does α have to satisfy to ensure that the system is stable?

The stability of the system depends on the poles, which are given by

$$\begin{aligned} p_1, p_2 &= \frac{-\alpha \pm \sqrt{\alpha^2 - 4\alpha^2}}{2} \\ &= \frac{-\alpha \pm j\alpha\sqrt{3}}{2} \end{aligned}$$

When the system is causal, all poles must be on the left of the jw-axis for the system to be stable. Thus, we require that $\alpha > 0$.

Problem 6.

(a) (**4 pts**) Let $x[n] = \left(-\frac{1}{2}\right)^n e^{j\frac{\pi}{4}n} u[n]$. Compute the Z-transform along with the associated ROC.

We have

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^n e^{j\theta n} z^{-n} \\ &= \sum_{n=0}^{\infty} \left(ae^{j\theta} z^{-1}\right)^n \\ &= \frac{1}{1 - ae^{j\theta} z^{-1}}, \end{aligned}$$

if $\left|ae^{j\theta} z^{-1}\right| < 1$. Thus, the ROC is specified by

$$|z| > |a|$$

(b) (**6 pts**) Consider a signal $x[n]$ given by

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n u[n] & n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Compute the Z-transform along with its ROC. Also indicate all zeros and poles.

$$\begin{aligned} X(z) &= \sum_{n=0,2,4,6,8,\ldots,\infty} \left(\frac{1}{2}\right)^n z^{-n} \\ &= \sum_{r=0,1,2,3,4,\ldots,\infty} \left(\frac{1}{2}\right)^{2r} z^{-2r} \\ &= \sum_{r=0}^{\infty} \left(\frac{1}{4} z^{-2}\right)^r \\ &= \frac{1}{1 - 0.25z^{-2}} \\ &= \frac{z^2}{z^2 - 0.25} \end{aligned}$$

if $\left|0.25z^{-2}\right| < 1$. Thus the ROC is specified by

$$|z| > 0.5$$

In addition, there are two zeros at $z = 0$, one pole at $z = 0.5$, and another pole at $z = -0.5$.