Fourier Series

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Where are we heading?

- For LTI systems, output is the convolution of input signal and impulse response

- Frequency domain descriptions such as Fourier series and Fourier transform provide an alternative view to analyze LTI systems with number of advantages.

  e.g.
  - Convolution is simplified in frequency domain.
  - It is intuitive and natural to analyze LTI systems.

- Representation of a signal in terms of basic signals

  Such set of basic signals should satisfy

  01 - We need to be able to represent any signal in terms of such set.

  02 - Response of an LTI system to any basic signal must be simple and insightful.

In short;

01 — Signal Representation

02 — System Characterization

Fortunately; complex exponentials form such set of basic signals.

Yes! Complex exponentials are eigenfunctions of LTI systems.

How?

- Consider a CT LTI system with impulse response $h(t)$ and input

$$x(t) = e^{st} \quad , s - \text{complex}$$

$x(t) = e$, s — complex.

output, $y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$

$$= \int h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st}\, H(s)$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$

Let's observe :-

input : e^{st}

output : $e^{st} H(s)$ (same signal scaled by $H(s)$)

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow e^{st} H(s)$$

$\Rightarrow$ e^{st} is an eigenfunction of LTI system with eigenvalue $H(s)$.

- DT LTI SYSTEM :-

  Impulse response ; $h[n]$

  Input signal ; $x[n] = z^n$ (Complex Exponential)

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \left(\sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \right) z^n$$

$$= H(z)\, z^n$$

where $H(z) \triangleq \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$

$\Rightarrow$ z^n is an eigenfunction of DT LTI system with eigenvalue $H(z)$.

<u>Q :</u> What is this all about ? Where are we heading?

Ans: Consider CT LTI system

Compare:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \qquad y(t) = \int x(\tau)\,h(t-\tau)\,d\tau$$

versus

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow \qquad y(t) = H(s)\,e^{st}$$

$\downarrow$

- No Complication
- No Convolution
- output is just a product.

<u>Example 01</u>

<u>01</u> $x(t) = e^{j2t}$ input to CT LTI system

with $H(2j) = 1 + j\sqrt{3} = 2e^{j\pi/6}$

$y(t) = x(t) * h(t) = ?$ We don't know $h(t)$ completely!

$y(t) = e^{j2t}\,H(2j) = 2\,e^{j(2t+\pi/6)}$

Simple!

<u>Example 02</u>

$$\longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow \quad y(t)$$

$x(t) = 4\cos(2t)$

$H_1(2j) = 1 + j\sqrt{3} = 2e^{j\pi/6}$

$H_1(-2j) = 1 - j\sqrt{3} = 2e^{-j\pi/6}$

$H_2(+2j) = 1 + j = \sqrt{2}\,e^{j\pi/4}$

$H_2(-2j) = 1 - j = \sqrt{2}\,e^{-j\pi/4}$

$$x(t) = \underbrace{2e^{2jt}}_{x_1(t)} + \underbrace{2e^{-2jt}}_{x_2(t)}$$

$$y_1(t) = 4\sqrt{2}\, e^{2jt + j\pi/6 + j\pi/4}$$

$$y_2(t) = 4\sqrt{2}\, e^{-2jt - j\pi/6 - j\pi/4}$$

$$y(t) = y_1(t) + y_2(t) = 8\sqrt{2}\cos(2t + \pi/6 + \pi/4)$$

☆ OBSERVATIONS :

- these examples are revealing something "IMPORTANT"
- For LTI systems connected in series, we need to convolve input and impulse response to get an output.
- For complex exponential input e^{st}, we only need to multiply input with complex gain $H(s)$.
- Great! Complex exponentials are easy to work with.
- But! What about more general signals?
- Among general signals, we begin with "Periodic Signals," and show that any periodic signal can be represented in terms of periodic complex exponentials. Such representation is known as "FOURIER SERIES".

• FOURIER SERIES _ CT PERIODIC SIGNAL

- Periodic Signal ; $x(t) = x(t+T)$

  $T \rightarrow$ fundamental period.

Using T, we define $\omega_0 = \frac{2\pi}{T}$ (Fundamental frequency)

- Consider a signal : $e^{j\omega_o t}$ (Periodic with period T)

Also, $e^{jk\omega_o t}$, $k \in \mathbb{Z}$ (Periodic with period T)

Also, their linear combination given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \quad \text{— (A)}$$

is periodic with period T.

- $k=0$ term is the DC term
- $k=1$, $k=-1$ are first harmonic
- $k=2$, $k=-2$ — second —
- k-th term has "FUNDAMENTAL" period $\frac{T}{k}$.

- Representation in (A) is known as FOURIER SERIES REPRESENTATION of a CT periodic signal.
- a_k, $k=0, \pm 1, \pm 2 \ldots$ are known as FOURIER SERIES COEFFICIENTS.

Q: Can we represent any periodic signal as in (A)?

ANS: Yes! (Almost, all of signals)

Then, we need to develop a method to find a_k given $x(t)$. Let's do a simple example.

Example :

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$

$$T = \frac{1}{2}, \quad \omega_o = 4\pi$$

$$x(t) = 2e^{j4\pi t} + 2e^{-j4\pi t} - \frac{3j}{2} e^{j8\pi t} + \frac{3j}{2} e^{-j8\pi t}$$

$\Rightarrow \quad a_1 = 2, \quad a_{-1} = 2$

$a_2 = -\frac{3j}{2}, \quad a_{-2} = \frac{3j}{2}$

$a_k = 0$ otherwise

- In this example we could read-off Fourier Series coefficients.
- What should we do for a general signal?

- Consider

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

(Integration over one period)

$$= \sum_{k=-\infty}^{\infty} a_k \int_T e^{jk\omega_0 t} e^{-jn\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k T \delta[n-k] = T a_n$$

SUMMARY:

We can represent any CT periodic signal $x(t)$ with fundamental period T and $\omega_0 \triangleq \frac{2\pi}{T}$ as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- This representation is called FOURIER SERIES REPRESENTATION OR SYNTHESIS EQUATION.
- "a_k" is called k-th FOURIER SERIES (FS) COEFFICIENT, and is given by

$$\frac{1}{T}\int x(t)\, e^{-jk\omega_0 t}\, dt$$

COEFFICIENT, and is given by

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Equation, known as ANALYSIS EQUATION, is used, in general, to compute k-th FS coefficient.