Fourier Series Examples, and Properties

Sunday, 3 February 2019 9:01 AM

We know so far;

SYNTHESIS EQUATION:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

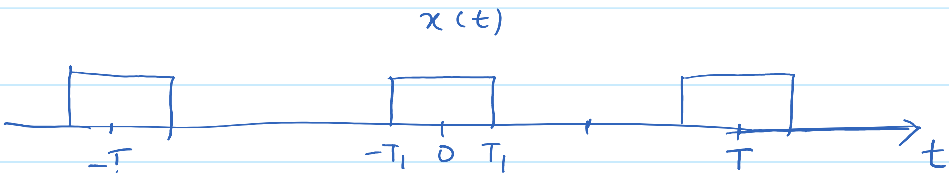
ANALYSIS EQUATION:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t}$$

* Note here that the integral in the analysis equation is over one period, that is, we can integrate over any one period time interval.

EXAMPLES:

- Example 1 : SQUARE WAVE



one period description of $x(t)$:

$$x(t) = \begin{cases} 1 & |t| \leq T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

We can find FOURIER SERIES COEFFICIENS As

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T\omega_0 k}\,\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{j}$$

$$= \frac{1}{2\pi k}\, 2\sin(k\omega_0 T_1)$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi} \quad \text{— } ①$$

To find a_0; we can either use ①

or determine, by definition,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \boxed{\frac{2T_1}{T}}$$

Using ①;

$$a_0 = \frac{0}{0}$$

Apply L'Hopital's Rule $\Rightarrow$ $a_0 = \dfrac{2T_1}{T}$

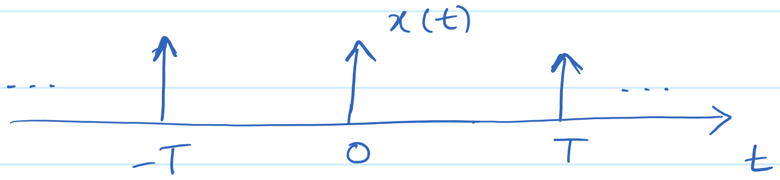
- To actually see how complex exponentials combine to form a square wave, see square-wave-FS.pdf, where

$$\sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t}$$

for $T_1 = \frac{T}{8}$, a_k given in (1) and different values of N.

<u>Example 2:</u> IMPULSE TRAIN

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} e^{-jk\omega_0 (0)} = \frac{1}{T}$$

<u>Interpretation:</u> All complex exponentials $\left\{ e^{jk\omega_0 t} \right\}_{k=-\infty}^{\infty}$ contribute equally to form an impulse train.

• PROPERTIES OF FOURIER SERIES.

Motivation :-

- As we see later, FS properties allow us to compute FS coefficients in a much simpler way.
- We want to study the effect of different transformations applied on signal on FS coefficients of the signal.

- For convenience, define a shorthand notation:

$$x(t) \xleftrightarrow{FS} a_k$$
$$y(t) \xleftrightarrow{FS} b_k$$
$$z(t) \xleftrightarrow{FS} c_k$$

• PROPERTIES:

01 - LINEARITY:

$$\alpha x(t) + \beta y(t) \xleftrightarrow{FS} \alpha a_k + \beta b_k$$

02 - TIME SHIFT:

$$x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$$

Proof:

Let $y(t) = x(t-t_0)$

By definition;

$$b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

$$b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

Let $t - t_0 = p$

$$\Rightarrow \quad b_k = \underbrace{\frac{1}{T}\int_T x(p)\, e^{-jk\omega_0 p}\, dp}_{a_k}\; e^{-jk\omega_0 t_0}$$

$$\Rightarrow \quad b_k = a_k \left(e^{-jk\omega_0 t_0} \right)$$

<u>Interpretation:</u> Time shift in the signal introduces phase change in FS coefficients. that is, magnitude of FS coefficients does not change with the time-shift of the signal.

03- TIME _ REVERSAL

Property: $x(-t) \longleftrightarrow a_{-k}$

<u>Proof</u> :

Let $y(t) = x(-t)$

$$b_k = \frac{1}{T}\int_T x(-t)\, e^{-jk\omega_0 t}$$

Let $-t \rightarrow p$

$$\Rightarrow \quad b_k = \frac{1}{T}\int_T x(p)\, e^{-j(-k)\omega_0 p}\, dp$$

$= a_{-k}$

Reversal in time-domain $\Rightarrow$ Reversal of FS coefficients.

04- CONJUGATE PROPERTY AND SYMMETRY

Property: $x^*(t) \longleftrightarrow a^*_{-k}$

Proof; Do it yourself!

Implications :-

For a real signal;

$$x(t) = x^*(t)$$

$$\Rightarrow \quad a_k = a^*_{-k}$$

This is known as conjugate-symmetry of FS coefficients of a real signal.

- Using conjugate symmetry property and time reversal property, we can show following properties.
  - If signal $x(t)$ is real and even.

$$x(t) = x^*(t) \quad \Rightarrow \quad a_k = a^*_{-k}$$

$$x(t) = x(-t) \qquad\qquad a_k = a_{-k}$$

$a_k \to$ Real and Even.

- If $x(t)$ is real and odd,

FS coefficients ; odd and purely imaginary

that is,

$$a_k = -a_{-k}, \qquad a_k = -a_k^*$$