Properties (Contd.), Examples, Convergence of Fourier Series

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We continue exploring properties first before looking at their usage in couple of examples

So far, we have studied linearity, time-shift time reversal, conjugation

- TIME-SCALING
  - Property : $x(\alpha t) \longleftrightarrow a_k \quad (\alpha > 0)$
  - Interpretation: FS coefficients do not change with time scaling.

    $x(\alpha t)$ is periodic with $\frac{T}{\alpha}$.
  - Proof :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha \omega_0)t}$$

- DIFFERENTIATION IN TIME

Property: $\frac{d}{dt}(x(t)) = x'(t) \longleftrightarrow jk\omega_0 a_k$

- Very easy to prove.

- INTEGRATION IN TIME

Property: $\displaystyle\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{a_k}{jk\omega_0}$

Conditions: $a_0 = 0$ required for $\displaystyle\int_{-\infty}^{t} x(\tau)\, d\tau$ to be finite valued and periodic.

Consequences; DC value information is lost during differentiation, and therefore cannot be recovered as a running integral.

- PARSEVAL'S RELATION
  - Relates energy of the signal and FS coefficients

Relation: $$\frac{1}{T}\int_{T} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Interpretation: Energy of a signal over one period is given by sum of absolute squared coefficients.

Parseval relation is useful in a sense that it allows to compute energy of a signal using its FS coefficients.

Proof: Assignment 03

Hint: Use orthogonality of complex exponentials.

- MULTIPLICATION PROPERTY

Property: $x(t)\, y(t) \longleftrightarrow \sum_{l=-\infty}^{\infty} a_l\, b_{k-l} = \sum_{l=-\infty}^{\infty} a_{k-l}\, b_l.$

Proof: Assignment 03

Interpretation: Multiplication of two signals in time domain corresponds to convolution in frequency domain (In general).

Since frequency domain here is described by FS coefficients, we have DT convolution of FS coefficients of $x(t)$ and $y(t)$.

- PERIODIC CONVOLUTION

- We first need to define periodic convolution as we are dealing with periodic signals.

- Periodic convolution of two CT periodic signals $x(t)$ and $y(t)$, each with same period T, is given by

$$z(t) = \int_T x(\tau)\, y(t-\tau)\, d\tau$$

- Difference here is that we have an integral over one period only.

- $z(t)$ is also periodic with period T.

- FS coefficients of $z(t)$ are given by

$$c_k = T a_k b_k \quad \text{(Convolution Property)}$$

<u>Proof</u>: Assignment 03.

<u>Interpretation</u>: Convolution in time-domain corresponds to multiplication of FS coefficients.

- This is dual (opposite) of the multiplication property.

- We have covered all properties of FS listed in Table 3.1 (Textbook, page 206).
- Remember these properties.
- We will not derive these properties again.
- However, we will reuse the properties / derivations.

- Let's look at some use of the properties of FS.
  - Example 3.6
  - Example 3.7
  - Example 3.8

- CONVERGENCE OF FOURIER SERIES

Earlier, we said that we can represent any CT periodic signal as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \begin{pmatrix} \text{Synthesis} \\ \text{Equation} \end{pmatrix}$$

where

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \quad \begin{pmatrix} \text{Analysis} \\ \text{Equation} \end{pmatrix}$$

Q: We can compute $\{a_k\}_{k=-\infty}^{\infty}$ for given $x(t)$ and synthesize a signal

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{-jk\omega_0 t}$$

Under what conditions

$$\lim_{N \to \infty} x_N(t) = x(t).$$

In other words, we need to know whether the Fourier Series representation converges to the signal OR we want to validate the equality in the synthesis equation.

We have two set of "somewhat" equivalent conditions which ensure the convergence of FS. Before stating the conditions, we define an error (point-wise)

$$e_N(t) \triangleq x(t) - x_N(t)$$

and average error per period as

and average error per period as

$$E_N = \int_T |e_N(t)|^2 dt$$

One set of Conditions :-

If $\int |x(t)|^2 dt < \infty$ (Finite energy over a period)

$$\lim_{N\to\infty} E_N = 0$$

that is the average error goes to zero and FS representation converges to the signal.

Note that the zero average error does not mean pointwise error is zero.

In other words $\lim_{N\to\infty} E_N = 0$ does not gaurantee that

$\lim_{N\to\infty} e_N(t) = 0$ | $\lim_{N\to\infty} x_N(t) = x(t)$

$\lim_{N\to\infty} E_N = 0$ Convergence in the mean

$\lim_{N\to\infty} e_N(t) = 0$ Point-wise Convergence

More strong notion of convergence.

Alternative set of Conditions :-

- Dirichlet proposed conditions of convergence which ensures that $x_N(t)$ converges to $x(t)$ everywhere as $N \to \infty$, except at the point of discontinuity where $\lim_{N\to\infty} x_N(t)$ converges to average value of the signal on either sides of point of discontinuity.

Condition 1 ; $\int_T |x(t)|\, dt < \infty$ (Absolute Integrable)

Condition 2 : $x(t)$ has finite number of discontinuities over one period

Condition 3 : $x(t)$ has finite number of maxima and minima over one period.