- Now we consider the representation of DT periodic signal in terms of complex exponentials

- Consider a DT periodic signal $x[n]$ with period $N$, that is

$$x[n+N] = x[n]$$

- In the representation of $x[n]$ using complex exponentials, only those complex exponentials will appear which are periodic with period $N$.

- We first define

$$w_0 \triangleq \frac{2\pi}{N}$$

- We studied DT complex exponentials of the form $e^{jkw_0n} = e^{jk\frac{2\pi}{N}n}$, $k = 0, \pm 1, \pm 2 \ldots$, $k \in Z$

which is a typical $N$-periodic complex exponential.

- What else do we know about

$$\left\{ e^{jkw_0n} \right\}, \quad k = 0, \pm 1, \pm 2 \ldots ?$$
$$k \in Z$$

- Do you remember that we only have $N$ unique complex exponentials in this set?

- Let us illustrate

For example, $N = 10$

we can have $k = 211$

$$e^{jk\omega_0 n} = e^{j211\left(\frac{2\pi}{10}\right)n} = \underbrace{e^{j42\pi n}}_{1} e^{j\left(\frac{2\pi}{10}\right)n}$$

$$= e^{j\left(\frac{2\pi}{10}\right)n}$$

$\downarrow$

$k = 1$

$\Rightarrow$ For $k = 211$ and $k = 1$, we, in fact, have same complex exponential.

- Let me illustrate graphically,

For $N = 4$;

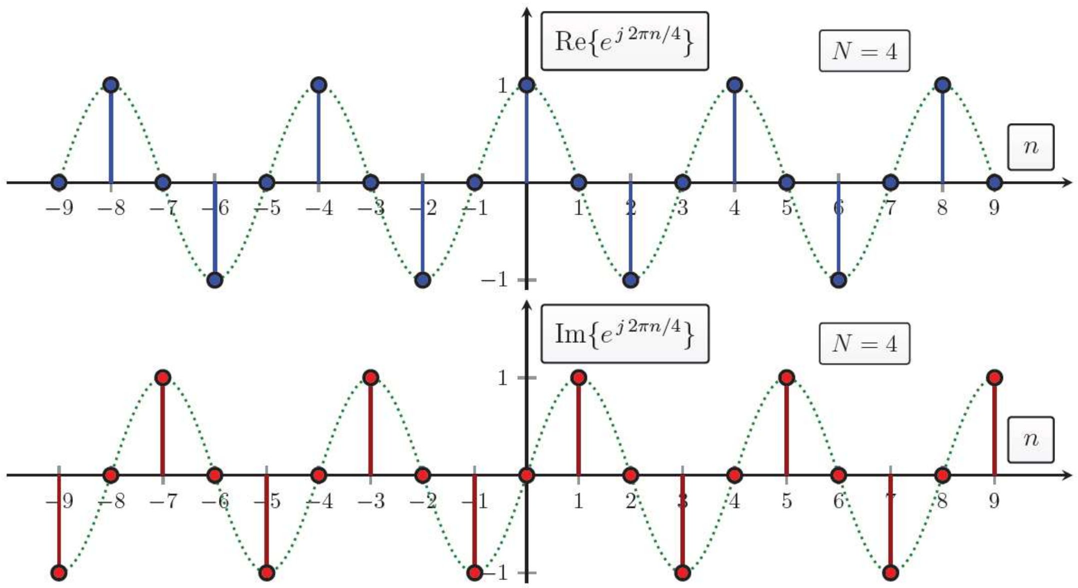
$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4}$$

- We plot below $e^{j\frac{2\pi}{4}n}$ and $e^{j5\frac{2\pi}{4}n}$

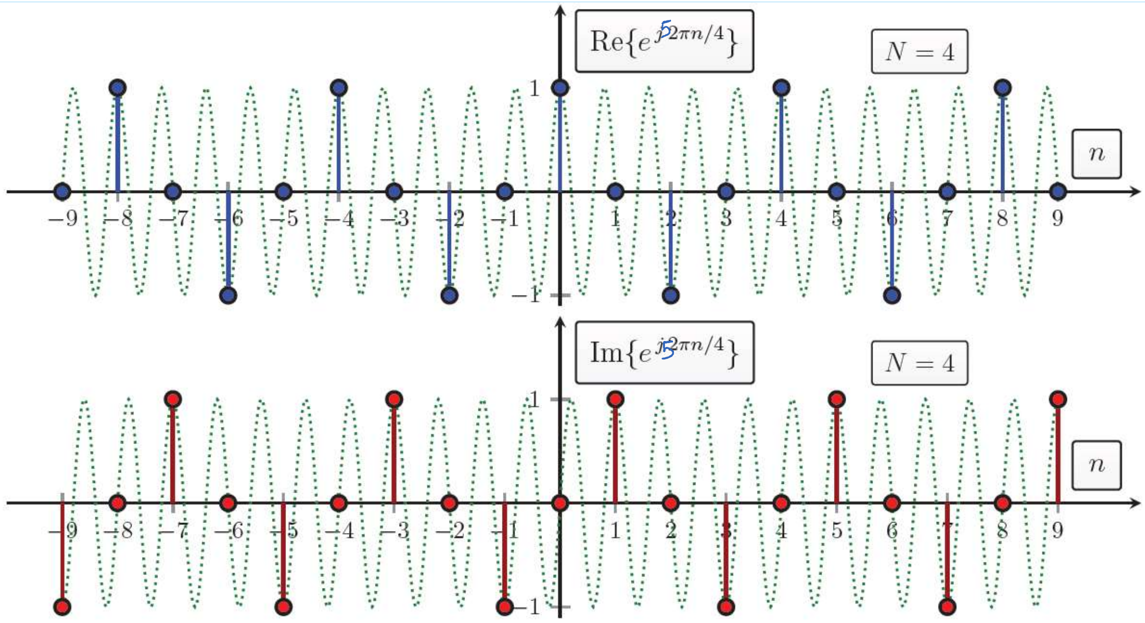
- Obviously; $e^{j\frac{2\pi}{4}n} = e^{j5\frac{2\pi}{4}n}$ integer $n$.

- For better understanding; we have also plotted signals for continuous $n$ as dashed lines

$\underline{e^{j\frac{2\pi}{4}n}}$ :-



$e^{j5\frac{2\pi}{4}n}$ :-




It is evident from above illustration that: when sampled, higher order frequencies are "aliased" to appear identical to lower order frequencies.

Consequently; A set of N-periodic complex exponentials

$$\left\{ e^{jk\omega_0 n} \right\} \quad , \quad k \in \mathbb{Z} \quad , \quad \omega_0 = \frac{2\pi}{N}$$

only have N unique complex exponentials, that is

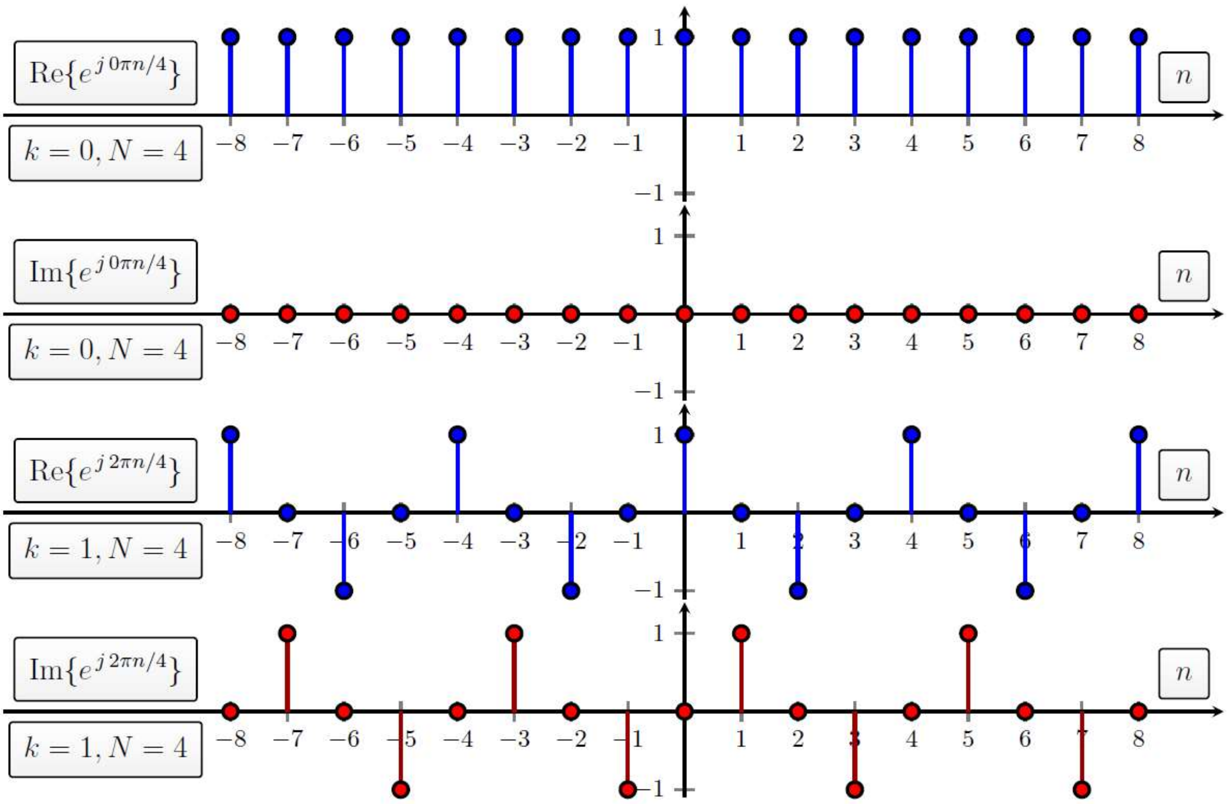
$$1, \; e^{j\omega_0 n}, \; e^{2j\omega_0 n}, \; \ldots, \; e^{j(N-1)\omega_0 n} .$$

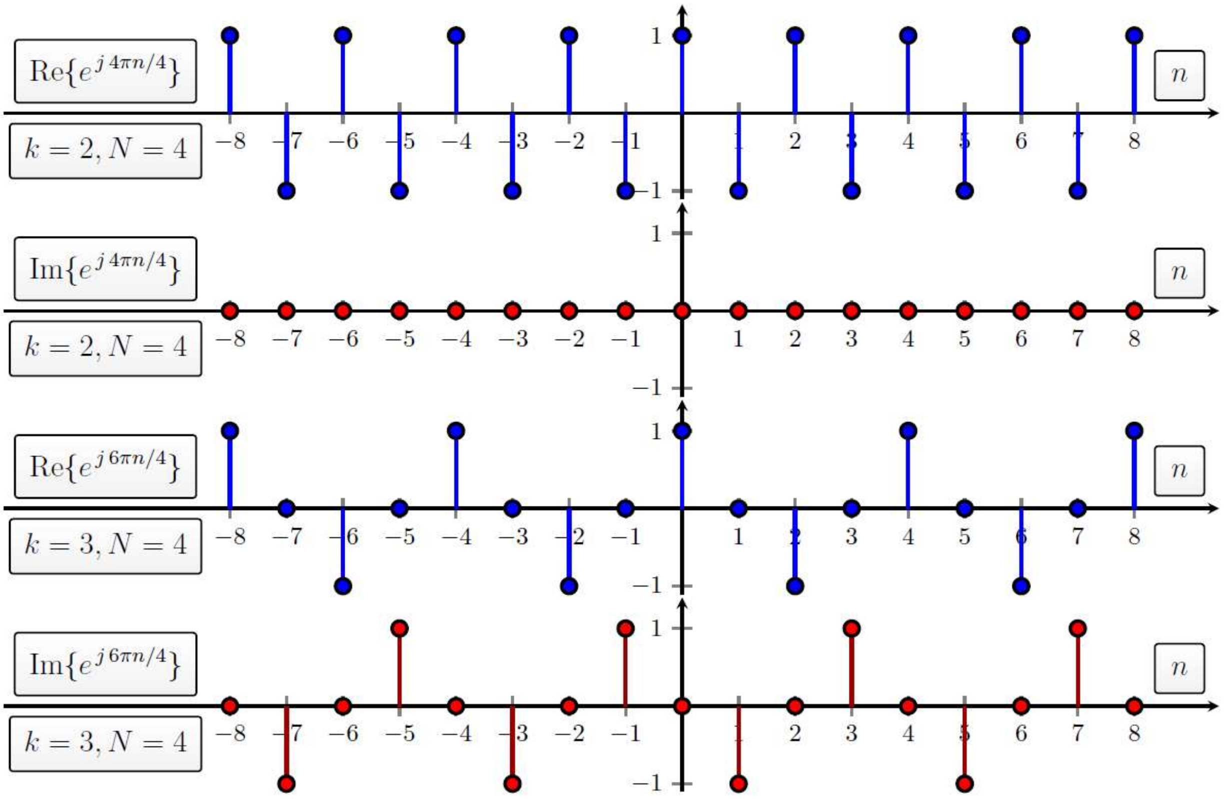
<u>Example</u>

For $N = 4$;

we have $e^{jk\frac{2\pi}{4}}$ , $k = 0, 1, 2, 3$

which is plotted below.



## One more viewpoint:

For some $N$; consider $e^{jk\frac{2\pi}{N}n}$

- Mathematically, $k$ and $n$ are swappable
- Define a function $F(n,k) = e^{jk\frac{2\pi}{N}n}$

$$\Rightarrow \quad F(n+N,k) = F(n,k) = F(n,k+N)$$

- For both $n$ and $k$, we only need to use $N$ values; the redundancy in $n$ is because complex exponential is periodic and redundancy in $k$ is because of frequency aliasing.

- **FOURIER SERIES REPRESENTATION:**

With the foundation laid above, we are all set to define FS representation of DT period signal.

<u>Definition:</u> Any DT periodic signal $x[n]$ with period $N$ can be represented as

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

Or equivalently

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} \qquad \text{(SYNTHESIS EQUATION)}$$

- Here $\langle N \rangle$ indicates that we need to take consecutive integer values of $k$ and we can start from anywhere.

- $\{a_k\}_{k=0}^{N-1}$ are FS coefficients which quantify the contribution of complex exponentials $\{e^{jk\omega_0 n}\}_{k=0}^{N-1}$

- Another viewpoint: $a_k = a_{k+N}$, that is FS coefficients are periodic.

Q: Given $x[n]$; how to determine $a_k$?

Simple answer: (Carefully) look at the synthesis equation.

- we have N linear equations and N unknowns. Use Matrix Algebra

Alternatively:

First we note the following orthogonality

$$\sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} = \sum_{n=\langle N\rangle} e^{jk\frac{2\pi}{N}n} = \begin{cases} N & k=0, \pm N, \pm 2N \dots \\ 0 & \text{otherwise} \end{cases}$$

Using this we can show that

$$\sum_{\langle n\rangle} x[n] e^{-jk\omega_0 n} = \sum_{n=\langle N\rangle} \left( \sum_{p=\langle N\rangle} a_p e^{jp\omega_0 n} \right) e^{-jk\omega_0 n}$$

$$= \sum_{p=0}^{N-1} a_p \left( \sum_{n=0}^{N-1} e^{j(p-k)\omega_0 n} \right)$$

$$= \boxed{N a_k}$$

OR

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n}$$ (ANALYSIS EQUATION)

STOP

- Develop some thoughts!
- What is going on?
- Compare DT and CT Analysis / Synthesis equations and description of FS coefficients.

- Ask yourself some fundamental questions and answer them. Obviously, you can come to see me if you could not come to the lecture.

- **EXAMPLES:**

Example 1: Sum of sinusoids

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

What is the period $N$? Look for $2\pi/N$ and the first term is slower. Period $N = 16$, which implies $\omega_0 = \pi/8$.

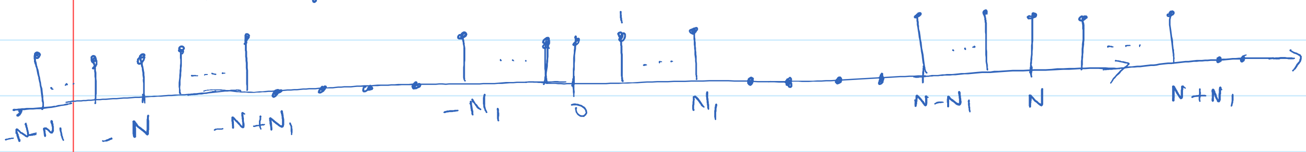
$$x[n] = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n}) + \frac{1}{2}(e^{j\pi/4}e^{j2\omega_0 n} + e^{-j\pi/4}e^{-j2\omega_0 n})$$

Hence

| | | | | | |
|---|---|---|---|---|---|
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | $a_{-18}$ | $a_{-17}$ | $a_{-16}$ | $a_{-15}$ | $a_{-14}$ |
| | $a_{-2}$ | $a_{-1}$ | $a_0$ | $a_1$ | $a_2$ |
| | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ | $a_{18}$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| Coefficient Values: | $\frac{1}{2}e^{j\pi/4}$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}e^{-j\pi/4}$ |

Example 2: DT Square wave

- $x[n]$, Period $N$, Width: $2N_1+1$, $2N_1+1 \le N$, Even
- Graphically:




- Mathematical description over time interval $[-N_1, N-N_1-1]$

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & N_1 < n < N-N_1 \end{cases}$$

- For rest of time; $x[n] = x[n+N]$

- Use analysis equation to find FS coefficients

$$a_k = \sum_{n=\langle N \rangle} x[n]\, e^{-jk\omega_0 n}$$

$$= \sum_{n=-N_1}^{N-N_1-1} x[n]\, e^{-jk\omega_0 n}$$

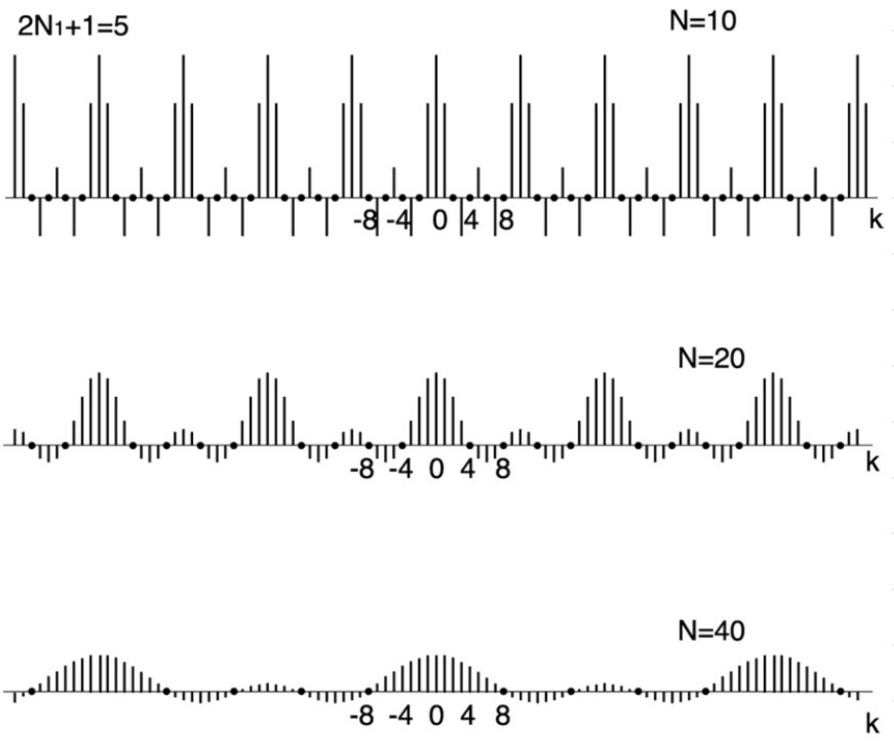
$$= \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \frac{e^{jk\omega_0 N_1}\left(1 - e^{-jk\omega_0(2N_1+1)}\right)}{1 - e^{-jk\omega_0}}$$

$$= \frac{1}{N} \frac{e^{jk\omega_0(N_1+\frac{1}{2})} - e^{-jk\omega_0(N_1+\frac{1}{2})}}{e^{jk\omega_0/2} - e^{-jk\omega_0/2}}$$

$$a_k = \frac{1}{N} \frac{\sin\left(k\omega_0\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\frac{k\omega_0}{2}\right)}$$

Periodic Sinc




- PROPERTIES OF FOURIER SERIES

- Most of the properties for DT FS are analogous to the properties of CT FS.

- For example;

If $x[n] \xleftrightarrow{FS} a_k$

$y[n] \xleftrightarrow{FS} b_k$

- Linearity : $\alpha x[n] + \beta y[n] \xleftrightarrow{FS} \alpha a_k + \beta b_k$

- Time reversal : $x[-n] \xleftrightarrow{FS} a_{-k}$

- Conjugation : $x^*[n] \longleftrightarrow a^*_{-k}$

- Using time reversal property and conjugation property, several symmetries can be shown for real, real even and real odd signals

- Time shift : $x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk\omega_0 n_0}$

These were obvious ones ; can be derived easily and we expect that you are now capable enough to derive these (quickly and easily).

There are less obvious ones like

Multiplication Property :

let

$z[n] = x[n]\, y[n]$

and $z[n] \longleftrightarrow c_k$

If you follow analogy ; $c_k$ would be convolution of $a_k$ and $b_k$, but here $a_k$ and $b_k$ are periodic in nature, which implies that $c_k$ is given by periodic convolution of $a_k$ and $b_k$, that is,

$$c_k = \sum_{<N>} a_l\, b_{k-l}$$

Convolution Property

Periodic convolution : $z[n] \triangleq \sum_{k=<N>} x[k]\, y[n-k]$

Let $z[n] \xleftrightarrow{FS} c_k$

Following analogy ; (or can be shown obviously)

$$\boxed{c_k = N\, a_k\, b_k}$$

- As we mentioned earlier, convolution property and multiplication properties are DUALS of each other.

- By DUAL or DUALITY (to be defined later), we mean that if time and frequency domains are swapped; the properties or behaviour of signals does not change.

## TIME SCALING :

- There is no direct property for compression of DT signal. Compression in DT corresponds to skipping signal. For example; compression by 2 means, you will be Taking samples at ... $-4, -2, 0, 2, 4$ ... . Since signal changes with skipping samples, there is no direct or simplified relation b/w compressed signal FS coefficients and original signal FS coefficients

- We do have relation for signal expansion. We first define signal expansion by a factor of $m$.

$$x_{(m)}[n] = \begin{cases} x[n/m] & n \text{ is a multiple of } m \\ 0 & \text{otherwise} \end{cases}$$

- Expanded signal; Insert $(m-1)$ zeros b/w samples.

- $x_{(m)}[n]$ is periodic with period $mN$.

- Q; what about FS coefficients of $x_{(m)}[n]$ ?

Let $x_{(m)}[n] \longleftrightarrow d_k$

$$d_k = \frac{1}{m} a_k$$

- Note that $a_k$ is periodic with period $N$, whereas $d_k$ is periodic with period $mN$.