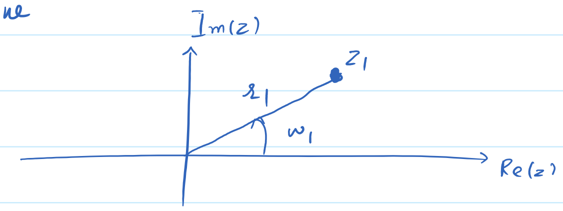
- <u>Z-Transform</u>: Generalization of DT Fourier transform.

Mathematically; Z-transform of a DT signal $x[n]$ is defined as

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \quad \text{— (1)}$$

for some complex variable z, which is expressed, in general, in polar form as $z = r e^{j\omega}$, where r represents the magnitude of complex z in complex plane from the origin and ω denotes the phase of the complex z.

Graphically; $z_1 = r_1 e^{j\omega_1}$ is shown below in complex plane



- RELATION WITH FOURIER TRANSFORM:

Using $z = r e^{j\omega}$, we rewrite $X(z)$ as

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(x[n]\, r^{-n}\right) e^{-j\omega n}$$

$\Rightarrow$ $X(re^{j\omega})$ is the DT FT of $x[n]\, r^{-n}$.

Alternatively;

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}} = X(z)\Big|_{|z|=1}$$

$\Rightarrow$ DT FT is the Z-transform evaluated over <u>unit circle</u> in the complex plane

$\Rightarrow$ Note the difference with the Laplace (s) transform.

- REGION OF CONVERGENCE (ROC)

By definition, ROC of the z-transform represents all values in z-plane for which the summation in (1) converges.

Example 1:

$$x[n] = \alpha^n u[n]$$

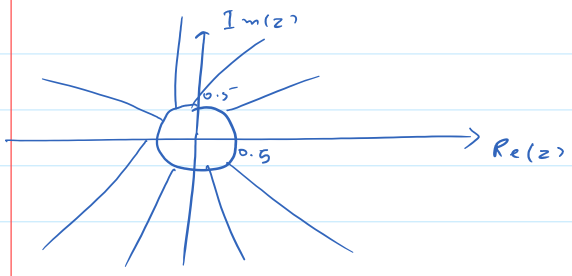
$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

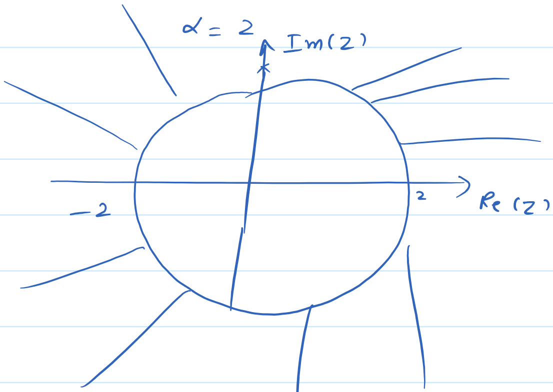
Summation converges if $|\alpha z^{-1}| < 1$ OR $|z| > |\alpha|$

$$\Rightarrow X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \qquad \text{ROC: } |z| > |\alpha|$$

ROC is shown below for $\alpha = 0.5$ and $\alpha = 2$

$\alpha = 0.5$





- When $\alpha = 0.5$, unit circle is part of ROC, which implies that FT of $(0.5)^n u[n]$ exists.

- When $\alpha = 2$, unit circle is not included in ROC
  $\Rightarrow$ FT of $(2)^n u[n]$ does not exist (converge)

Example 2:

$$x[n] = -\alpha^n u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{\infty} \alpha^n u[-n-1] z^{-n}$$

$$= -\sum_{p=1}^{\infty} (\alpha^{-1} z)^p = \frac{-\alpha^{-1} z}{1-\alpha^{-1} z} \qquad (n = -p)$$

$$= -\sum_{p=1}^{\infty} \left(\alpha^{-1}z\right)^{l} = \frac{-\alpha^{-1}z}{1-\alpha^{-1}z} \qquad (n=-p)$$

$$= \frac{z}{z-\alpha} \qquad \text{ROC: } |\alpha^{-1}z| < 1$$

$$|z| < |\alpha|$$

$\Rightarrow$ ROC is inside of circle of radius $|\alpha|$

Observations (Examples 1 and 2)

- For different signals, we can have same z-transform.

- Consequently, z-transform together with ROC uniquely identifies / describes a time-domain signal.

Example 3 :-

$$x[n] = \beta^{|n|} \qquad |\beta| > 0$$

$$x[n] = \beta^{n} u[n] + \beta^{-n} u[-n-1]$$

Using results of examples 1 and 2, and noting linearity of z-transform

$$X(z) = \frac{1}{1-\beta z^{-1}} - \frac{1}{1-\beta^{-1} z^{-1}} \qquad \text{ROC: } \beta < |z| < \frac{1}{\beta}$$

$0 < \beta < 1$; ROC is a ring of outer radius $\frac{1}{\beta}$ and inner radius β

For $\beta > 1$; $X(z)$ does not converge for any value of z.

- PROPERTIES OF ROC :
  - We now review the properties of ROC for z-transform.
  - We will develop some intuitive understanding of these properties
  - We will not formally derive these properties.

PROPERTY 1 :- ROC consists of a ring in the z-plane with origin as center.

Why; By definition ROC includes $z = re^{jw}$

- if $\sum_{n=-\infty}^{\infty} x[n]\, r^{-n}\, e^{-jwn}$ converges

OR

- FT of $x[n] r^{-n}$ converges

OR

- $x[n]\, r^{-n}$ is absolute summable, that is,

$$\sum_{n=-\infty}^{\infty} \left| x[n]\, r^{-n} \right| < \infty$$

- This condition only includes r (magnitude of z) and not w (Angle of z)
- Consequently, property 1 holds.

PROPERTY 2 :-

If $x[n]$ is of finite duration, ROC consists of entire plane except possibly $z=0$ and/or $z=\infty$.

* Note the difference here with Laplace transform.
* This difference requires explanation:

Consider a finite duration signal $x[n] = \begin{cases} \text{Non-zero} & N_1 \le n \le N_2 \\ 0 & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

Key-point:

Summation may include $-ve$ and $+ve$ powers of z.

- For $-ve$ power of z, summation does not converge for $z=0$
- For $+ve$ ——— $z=\infty$.

Different Cases:

- If N_1 is negative and N_2 is positive;

Both $z=0$ and $z=\infty$ are not included in ROC

- If N_1 is zero or positive; (Signal is causal)
  ROC includes $z = \infty$ and does not include $z = 0$

- If N_2 is zero or negative; (Signal is anti-causal)
  ROC includes $z = 0$ and does not include $z = \infty$

Examples :

Consider $x[n] = \delta[n]$ (Impulse)

$X(z) = 1$ ROC: Entire z-plane

For $x[n] = \delta[n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-1]\, z^{-n} = z^{-1} \qquad \text{ROC: } z \neq 0$$

For $x[n] = \delta[n+1]$

$$X(z) = z, \qquad \text{ROC: } z \neq \infty$$

Generalizing

For $x[n] = \delta[n-n_0]$

$$X(z) = z^{-n_0} \qquad \text{ROC: } \begin{cases} z \neq 0 & n_0 > 0 \\ z \neq \infty & n_0 < 0 \\ \text{Entire z-plane} & n_0 = 0 \end{cases}$$

PROPERTY 3 :

- If $x[n]$ is right-sided sequence and circle $|z| = r_0$ is included in ROC, then all finite z with $|z| > r_0$ are also included in ROC

- Furthermore, if $x[n]$ is causal, $z = \infty$ will also be included in ROC (Makes sense, following property 2)

- Evidence : Example 1

PROPERTY 4 :

- If $x[n]$ is left sided and $|z| = r_0$ circle is in ROC, all z with $0<|z|<r_0$ are also included in ROC.

- If $x[n]$ is anticausal; $z=0$ is also included in ROC.
- Evidence; Example 2

PROPERTY 5:

- If $x[n]$ is two sided and ROC includes circle $|z| = r_0$, ROC consists of a ring in z-plane that includes the circle $|z| = r_0$.
- Evidence: Example 3.

\# These 5 properties hold for ROC for any $X(z)$. We now review 3 more properties which hold for rational $X(z)$. These properties are infact special cases of first 5 properties.

PROPERTY 6:

- If $X(z)$ is rational, associated ROC does not contain any poles.
- Very simple to follow as poles, by definition, denote the points in z-plane where $X(z)$ does not converge

PROPERTY 7:-

- Generalizing property 6, ROC for rational $X(z)$ is bounded by poles or extends to infinity.

PROPERTY 8: (Combine property 7 and property 3)

- If $x[n]$ is right sided; ROC consists of the region outside of outer-most (farthest from origin) pole.

- Furthermore, if $x[n]$ is also causal, ROC includes $z = \infty$.

PROPERTY 9 (Combining properties 7 and 4)

- If $x[n]$ is left sided; ROC is the region inside of inner-most (nearest to the origin) pole and may include $z = 0$.
- If $x[n]$ is anticausal in addition, $z=0$ also included in ROC.

• INVERSE Z-TRANSFORM :-

Inverse Z-transform is mathematically defined as

$$x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$$

- Since z is complex, we have contour integration involved in the evaluation of Z-transform.

- In this course, we will not focus on Contour integration.
- We rather determine the inverse Z-transform using the properties of ROC, properties of Z-transform and use of well-known pairs OR using the power series expansion.

- POWER SERIES EXPANSION; By expanding $X(z)$ as

$$X(z) = \sum_{k=-\infty}^{\infty} \alpha_k z^{-k},$$

we can find inverse Z-transform as

$$x[n] = \sum_{k=-\infty}^{\infty} \alpha_k \delta[n-k]$$

See Examples: 10.9 – 10.13

- ANALYSIS OF DT LTI SYSTEMS USING Z-TRANSFORM

- We now look at use of z-transform in analyzing properties of LTI systems, e.g., stability and causality.

- Consider an LTI system with impulse response $h[n]$.
- Let $H(z)$ be the z-transform of $h[n]$ with ROC denoted by R.
- Given $H(z)$ and ROC, we seek to determine whether the system is causal and/or stable.

- CAUSALITY :-

- For DT LTI system to be causal, we require

$$h[n] = 0 \quad \text{for} \quad n < 0$$

- For $h[n]$ satisfying above condition, we have $H(z)$ given by

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

- Equivalently, $H(z)$ does not contain any +ve powers of z, which implies that $z = \infty$ is in ROC.

- Summarizing, DT LTI system is a causal system if ROC of z-transform of its impulse response is the exterior of a circle, including infinity.

Implies $h[n] = 0 \;\forall n < 0$

circle is origin; if $h[n]$ is of finite duration.

- If $H(z)$ is rational, the above condition holds iff
  - ROC is the exterior of a circle outside the outermost pole

including infinity (using properties).

- Since $z = \infty$ is included in ROC,

$\lim_{z \to \infty} X(z)$ converges, or equivalently, degree of numerator polynomial of $X(z)$ is less than or equal to the degree of denominator polynomial.

- **STABILITY:**

- For DT LTI system to be stable, we have

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Remember, this is also the condition of convergence of DT FT.

- Consequently, DT LTI system is stable, if DT FT of impulse response of the system converges.

- DT FT of $h[n]$ converges if ROC of $H(z)$ includes unit circle ($z = e^{j\omega}$).

- Summarizing; DT LTI system is stable if ROC of z-transform of impulse includes unit circle.

- Combining Causality and Stability:

- DT LTI system is stable and causal if
  - $z = \infty$ included in ROC (Causality)
  - $z = e^{j\omega}$ (unit circle) is included in ROC.

- For rational $H(z)$, system is causal and stable if all poles lie inside the unit circle.

See examples 10.21 – 10.24

- Good Luck !