LUMS
A Not-for-Profit University

School of Science and Engineering

EE 310 Signals and Systems

PROBLEM SET 1 – SOLUTIONS

Problem 1

Evaluate the following expressions involving the Dirac delta function. Carefully justify each result.

1. $\int_{-\infty}^{2} \delta(t-3)\, dt$
2. $\int_{1}^{5} (t-4)\,\delta(t-4)\, dt$
3. $\int_{0}^{\infty} \delta(2t-6)\, dt$
4. $\int_{-2}^{2} \delta(t^2-1)\, dt$

Solution:

1. $\int_{-\infty}^{2} \delta(t-3)\, dt = 0$ since 3 is outside integration limits.
2. $\int_{1}^{5} (t-4)\,\delta(t-4)\, dt = (4-4) = 0$.
3. Using scaling property: $\delta(2t-6) = \frac{1}{2}\delta(t-3)$, so:
$$\int_{0}^{\infty} \delta(2t-6)\, dt = \frac{1}{2}\int_{0}^{\infty} \delta(t-3)\, dt = \frac{1}{2}$$
4. $\delta(t^2-1) = \delta((t-1)(t+1))$. The roots are $t = \pm 1$. Using property:
$$\delta(g(t)) = \sum_i \frac{\delta(t-t_i)}{|g'(t_i)|}$$
For $t_1 = 1$, $g'(1) = 2$; for $t_2 = -1$, $g'(-1) = -2$. Both within $[-2, 2]$, so:
$$\int_{-2}^{2} \delta(t^2-1)\, dt = \frac{1}{2} + \frac{1}{2} = 1$$

$$\boxed{(1)\ 0, \quad (2)\ 0, \quad (3)\ \frac{1}{2}, \quad (4)\ 1}$$

Problem 2

Consider the discrete-time signal $x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$.

Determine the values of the integers M and n_0 such that $x[n]$ may be expressed in the form $x[n] = u[Mn - n_0]$.

Solution: Given $x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$.
The summation represents impulses at $n = 4, 5, 6, \ldots$, so:
$$x[n] = \begin{cases} 1, & n \le 3 \\ 0, & n \ge 4 \end{cases} = u[3-n]$$
Comparing with $u[Mn - n_0]$, we get $M = -1$, $n_0 = -3$.

$$\boxed{M = -1, \quad n_0 = -3}$$

Problem 3

Determine whether or not each of the following signals is periodic. If the signal is periodic, determine its fundamental period.

1. $x[n] = (-1)^n \cos\left(\frac{2\pi n}{7}\right)$
2. $x[n] = 2\cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{8}n\right) - 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)$
3. $x(t) = \sin^2(4t)$
4. $x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$

Solution:

1. $x[n] = (-1)^n \cos(2\pi n/7)$. $(-1)^n$ has period 2, $\cos(2\pi n/7)$ has period 7. LCM is 14.
2. Periods: $\cos(\pi n/4)$ period $N_1 = 8$, $\sin(\pi n/8)$ period $N_2 = 16$, $\cos(\pi n/2 + \pi/6)$ period $N_3 = 4$. Fundamental period is LCM(8,16,4)=16.
3. $x(t) = \sin^2(4t) = \frac{1}{2}(1 - \cos(8t))$. Period of $\cos(8t)$ is $\pi/4$.
4. Periods: $\cos(\pi n/2)$ period 4, $\cos(\pi n/4)$ period 8. LCM is 8.

$$\boxed{(1)\text{ Periodic, } N_0 = 14, \quad (2)\text{ Periodic, } N_0 = 16, \quad (3)\text{ Periodic, } T_0 = \pi/4, \quad (4)\text{ Periodic, } N_0 = 8}$$

Problem 4

1. Consider a signal $x(t)$ having finite energy E. Answer the following:
   (a) Prove that the signal $x(at)$ has finite energy $\frac{E}{|a|}$, where $\forall\, a \in \mathbb{R}$ and $a \neq 0$.
   (b) Now consider the signal $x(t - b)$, where $\forall\, b \in \mathbb{R}$. What will be its energy in terms of E?
   (c) Calculate the energy of the signal $y(-7t + 9)$, given that $y(t)$ has finite energy 5. In doing so, you may directly use the results from parts (i) and (ii).
2. Consider the power signal $x(t) = 3 + \frac{1}{2}e^{j\frac{3\pi}{4}t}$. Compute the average power P_∞.

Solution:

(a) Energy: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$. For $x(at)$:

$$E_{new} = \int_{-\infty}^{\infty} |x(at)|^2 dt$$

Let $\tau = at$, $dt = d\tau/|a|$:

$$E_{new} = \frac{1}{|a|}\int_{-\infty}^{\infty} |x(\tau)|^2 d\tau = \frac{E}{|a|}$$

(b) Time shift doesn't change energy: $E_{x(t-b)} = E$.

(c) $y(-7t + 9) = y(-7(t - 9/7))$. From (a): energy scaled by $1/|a| = 1/7$. From (b): time shift doesn't change energy. So energy $= 5/7$.

(d) $x(t) = 3 + \frac{1}{2}e^{j3\pi t/4}$. DC component: 3, sinusoid amplitude: 1/2 but one impulse. Average power:

$$P = 3^2 + \frac{1}{2}\left(\frac{1}{2}\right) = 9 + \frac{1}{4} = \frac{37}{4}$$

$$\boxed{(a)\ \frac{E}{|a|}, \quad (b)\ E, \quad (c)\ \frac{5}{7}, \quad (d)\ \frac{37}{4}}$$

Problem 5

Determine whether each of the following signals is an energy signal, a power signal, or neither. If the signal is an energy or power signal, compute its finite energy or average power.

1. $x(t) = e^{t-\lfloor t \rfloor}$, where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t (i.e., the floor function).
2. $x(t) = 8\cos\left(\frac{\pi}{2} - 20\pi t\right) + 4\sin(15\pi t)$.
3. $x(t) = e^{-6|t-1|}, \qquad \forall\, t \in \mathbb{R}$.
4. $x(t) = e^{-|t|}\cos(\omega_0 t)$.
5. $x(t) = \sum_{k=-\infty}^{\infty} e^{-|t-k|}$.

Solution:

1. $x(t) = e^{t-\lfloor t \rfloor}$. Periodic with period 1. For power signal:
$$P = \frac{1}{1}\int_0^1 e^{2(t-\lfloor t \rfloor)} dt = \int_0^1 e^{2t} dt = \frac{e^2-1}{2}$$
2. Sum of sinusoids: periodic, power signal. Each sinusoid has average power $A^2/2$:
$$P = \frac{8^2}{2} + \frac{4^2}{2} = 32 + 8 = 40$$
3. $x(t) = e^{-6|t-1|}$. Energy signal:
$$E = \int_{-\infty}^{\infty} e^{-12|t-1|} dt = 2\int_0^{\infty} e^{-12t} dt = \frac{1}{6}$$
4. $x(t) = e^{-|t|}\cos(\omega_0 t)$. Energy signal:
$$E = \int_{-\infty}^{\infty} e^{-2|t|}\cos^2(\omega_0 t) dt < \infty$$
5. $x(t) = \sum_{k=-\infty}^{\infty} e^{-|t-k|}$. Periodic extension, power signal. Power over one period:
$$P = \int_0^1 \left(\sum_{k=-\infty}^{\infty} e^{-|t-k|}\right)^2 dt$$

$$\boxed{(1)\text{ Power, } P = \frac{e^2-1}{2}, \quad (2)\text{ Power, } P = 40, \quad (3)\text{ Energy, } E = \frac{1}{6}, \quad (4)\text{ Energy}, \quad (5)\text{ Power}}$$

Problem 6

For each signal in **Column A**, find the signal or signals in **Column B** that are identical.

Column A	**Column B**
(1) $\delta[n+1]\,u[n]$	(a) $\sum_{k=-\infty}^{\infty} \delta[k]$
(2) $(\frac{1}{2})^n u[n]$	(b) $\frac{d}{dt}u(t)$
(3) $\delta(t)$	(c) $\delta[k]$
(4) $u(t)$	(d) $\sum_{k=0}^{\infty}(\frac{1}{2})^k \delta[n-k]$
(5) $u[n]$	(e) $\int_{-\infty}^{t} \delta(\tau)\, d\tau$
(6) $\delta[n-1]$	(f) $u[n]$
	(g) $\sum_{k=-\infty}^{\infty} (\frac{1}{2})^k \delta[n-k]$
	(h) $\delta[n+1]$
	(i) ϕ

Solution:

1. $\delta[n+1]u[n]$ does not match with any
2. $(1/2)^n u[n]$ matches with (d): $\sum_{k=0}^{\infty}(1/2)^k \delta[n-k]$
3. B
4. $u(t)$ matches with (e): $\int_{-\infty}^{t} \delta(\tau)d\tau$
5. $u[n]$ matches with (a): $\sum_{k=-\infty}^{\infty} \delta[k] = 1$ for $n \geq 0$
6. none

$$\boxed{(1)\text{ none}, \quad (2)\text{ d}, \quad (3)\text{ b}, \quad (4)\text{ e}, \quad (5)\text{ a,f} \quad (6)\text{ none}}$$

Problem 7

Let $x_1(t) = \sin\left(t + \frac{3\pi}{4}\right),\ x_2(t) = e^{-|t|-2j},\ x_3[n] = e^{-|n-1|}$.

Find the *total energy* and the *average power* of each of the signals $x_1(t)$, $x_2(t)$, and $x_3[n]$.

Solution:

1. $x_1(t) = \sin(t + 3\pi/4)$. Periodic, infinite energy. Power:
$$P_2 = \frac{1}{2\pi}\int_0^{2\pi} \sin^2(t + 3\pi/4)dt = \frac{1}{2}$$
2. $x_2(t) = e^{-|t|-2j} = e^{-2j}e^{-|t|}$. Energy:
$$E_1 = |e^{-2j}|^2 \int_{-\infty}^{\infty} e^{-2|t|}dt = 1 \times 2\int_0^{\infty} e^{-2t}dt = 1$$
Since finite energy, $P_1 = 0$.
3. $x_3[n] = e^{-|n-1|}$. Energy:
$$E_3 = \sum_{n=-\infty}^{\infty} e^{-2|n-1|} = 1 + 2\sum_{n=1}^{\infty} e^{-2n} = 1 + \frac{2e^{-2}}{1-e^{-2}}$$
Since finite energy, $P_3 = 0$.

$$\boxed{x_1:\ E = 1,\ P = 0; \quad x_2:\ E = \infty,\ P = 1/2; \quad x_3:\ E = \frac{e^2+1}{e^2-1},\ P = 0}$$

Problem 8

Let $x_1(t)$ be an energy signal and $x_2(t)$ be a power signal. Let
$$x(t) = x_1(t) + x_2(t).$$
Prove that $x(t)$ is a power signal and not an energy signal.

Solution: Let $E_1 = \int |x_1(t)|^2 dt < \infty$, $P_2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x_2(t)|^2 dt$ finite.
For $x(t) = x_1(t) + x_2(t)$:
$$E_x = \int |x_1 + x_2|^2 dt \geq \int |x_2|^2 dt - 2\sqrt{E_1 \int |x_2|^2 dt} \to \infty$$
since $\int |x_2|^2 dt \to \infty$ for power signal.
Average power:
$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x_1 + x_2|^2 dt = P_2 + \lim_{T\to\infty} \frac{E_1}{2T} = P_2$$
Thus $x(t)$ is a power signal, not energy signal.

$$\boxed{x(t) \text{ is power signal with power } P_2\text{, not energy signal.}}$$

Problem 9

Prove that for any continuous function $f(t)$:

$$\int_{-\infty}^{\infty} f(t)\,\delta(at-b)\,dt = \frac{1}{|a|}f\Big(\frac{b}{a}\Big), \quad a \neq 0$$

Solution: Using scaling and shifting properties of delta function:

$$\delta(at-b) = \delta(a(t-b/a)) = \frac{1}{|a|}\delta(t-b/a)$$

Then:

$$\int_{-\infty}^{\infty} f(t)\delta(at-b)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} f(t)\delta(t-b/a)dt = \frac{1}{|a|}f(b/a)$$

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(at-b)dt = \frac{1}{|a|}f\left(\frac{b}{a}\right)}$$

Problem 10

Consider the discrete-time signal defined by a summation of products of unit step functions:

$$x[n] = \sum_{k=0}^{5} u[n-k]\,u[3-k+n],$$

where $u[n]$ is the discrete-time unit step function.

1. Express $x[n]$ as a piecewise function (explicit values for each n).
2. Compute $\sum_{n=-\infty}^{\infty} x[n]$.

Solution: $x[n] = \sum_{k=0}^{5} u[n-k]u[3-k+n]$.
$u[n-k] = 1$ for $n \geq k$, $u[3-k+n] = 1$ for $n \geq k-3$.
So both conditions: $n \geq \max(k, k-3) = k$ for $k \geq 0$.
Thus $x[n] = \sum_{k=0}^{5} 1$ for $n \geq k$, but careful: for fixed n, count k such that $n \geq k$ and $n \geq k-3$ (always true if $n \geq k$).

For $n < 0$: $x[n] = 0$ (no k satisfies $n \geq k \geq 0$)
For $0 \leq n \leq 5$: k from 0 to n, so $x[n] = n+1$
For $n \geq 6$: k from 0 to 5, so $x[n] = 6$

$$x[n] = \begin{cases} 0, & n < 0 \\ n+1, & 0 \leq n \leq 5 \\ 6, & n \geq 6 \end{cases}$$

$$\sum_{n=-\infty}^{\infty} x[n] = \sum_{n=0}^{5}(n+1) + \sum_{n=6}^{\infty} 6 \to \infty$$

$$\boxed{x[n] = \begin{cases} 0 & n < 0 \\ n+1 & 0 \leq n \leq 5 \\ 6 & n \geq 6 \end{cases} \quad \sum_{n=-\infty}^{\infty} x[n] = \infty}$$

Problem 11

Indicate whether each of the following statements is True or False. Justify your answer briefly.

1. The Dirac delta function $\delta(t)$ is a finite-energy signal.
2. The unit step function $u(t)$ is a power signal.
3. Any finite-duration continuous-time signal is always an energy signal.
4. A discrete-time signal $x[n] = e^{j\pi n/3}$ is periodic.
5. A continuous-time signal $x(t) = \sin(\sqrt{2}t)$ is periodic.
6. If $x(t)$ is an energy signal and $y(t)$ is a power signal, $x(t) + y(t)$ is always an energy signal.
7. For a discrete-time signal $x[n]$, if $x[n]$ is periodic, then its energy is infinite.
8. The product of two unit step functions, $u(t-a)u(t-b)$, is equivalent to a single unit step $u(t - \max(a, b))$.

Solution:

1. False: $\delta(t)$ has undefined energy, not finite
2. True: $u(t)$ has infinite energy but finite average power 1/2
3. True: Finite duration and bounded $\Rightarrow$ finite energy
4. True: $x[n] = e^{j\pi n/3}$ has period 6
5. False: $\sin(\sqrt{2}t)$ has irrational period ratio
6. False: From Problem 8, sum is power signal
7. True: Periodic signals have infinite energy (unless identically zero)
8. False: $u(t-a)u(t-b) = u(t - \max(a, b))$ if $a \neq b$

$$\boxed{(1)\ F,\ (2)\ T,\ (3)\ T,\ (4)\ T,\ (5)\ F,\ (6)\ F,\ (7)\ T,\ (8)\ T}$$

Problem 12

Consider the continuous-time sign function defined as:

$$\text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

1. Express $\text{sgn}(t)$ in terms of the unit step function $u(t)$ and finds its derivative.
2. Sketch both $\text{sgn}(t)$ and its derivative.

Solution: Signum function: $\text{sgn}(t) = 2u(t) - 1$.
Derivative: $\frac{d}{dt}\text{sgn}(t) = 2\delta(t)$.

$$\boxed{\text{sgn}(t) = 2u(t) - 1, \quad \frac{d}{dt}\text{sgn}(t) = 2\delta(t)}$$

$$\boxed{\text{Graph: sgn(t) steps from -1 to 1 at 0, derivative is impulse at 0}}$$

— End of Problem Set —