LUMS
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School of Science and Engineering

EE 310 Signals and Systems

PROBLEM SET 7

Key Concepts of the Discrete-Time Fourier Transform (DTFT)

The **Discrete-Time Fourier Transform (DTFT)** serves as the frequency-domain representation of a discrete-time signal. It allows us to decompose a sequence into its constituent complex exponential components.

The relationship between the time-domain sequence $x[n]$ and its frequency-domain representation $X(e^{j\omega})$ is defined by the following two equations:

- **Analysis Equation (Forward DTFT):** Converts the time-domain signal into the frequency domain.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Synthesis Equation (Inverse DTFT):** Reconstructs the time-domain signal from its frequency components.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

A critical distinction of the DTFT is its **periodicity**. Because $e^{-j(\omega+2\pi)n} = e^{-j\omega n} \cdot e^{-j2\pi n} = e^{-j\omega n}$ for any integer n, the spectrum repeats every 2π radians. Consequently, we typically analyze the DTFT over the fundamental interval $\omega \in [-\pi, \pi]$.

DTFT Properties

These properties allow us to manipulate signals in the time domain by performing simpler operations in the frequency domain.

Property	**Time Domain** $x[n]$	**Frequency Domain** $X(e^{j\omega})$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Convolution	$x[n] * h[n]$	$X(e^{j\omega})H(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's Relation	$\sum \lvert x[n]\rvert^2$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

Common DTFT Pairs

The following pairs are frequently used to solve convolution and system response problems without evaluating the summation directly.

Signal Name	**Time Domain** $x[n]$	**Frequency Domain** $X(e^{j\omega})$
Unit Impulse	$\delta[n]$	1
Shifted Impulse	$\delta[n-n_0]$	$e^{-j\omega n_0}$
Exponential Decay	$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$
Rectangular Pulse	$u[n]-u[n-N]$	$e^{-j\omega\frac{N-1}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$
Ideal Lowpass	$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert \leq \omega_c \\ 0, & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
Constant (DC)	1 (for all n)	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

Convergence and Physical Interpretation

The DTFT converges if $x[n]$ is absolutely summable. The magnitude $|X(e^{j\omega})|$ describes the "strength" of various frequency components, while the phase $\angle X(e^{j\omega})$ captures the relative alignment (delay) of these frequencies. This preamble provides the theoretical foundation for the analytical problems presented in this set.

Problem 1

The DTFT is defined as: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

(a) Prove that $X(e^{j\omega})$ is periodic with period 2π. Explain why this periodicity exists.

(b) Given $x[n] = \delta[n]$, compute the DTFT analytically and verify periodicity.

Problem 2

Prove uniqueness: if $X_1(e^{j\omega}) = X_2(e^{j\omega})$ for all $\omega \in [-\pi, \pi]$, then $x_1[n] = x_2[n]$ for all n. What convergence conditions ensure the DTFT exists?

Problem 3

Frequency-Shift (Modulation) Property

(a) Prove: if $x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega})$, then $e^{j\omega_0 n}x[n] \xrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)})$

(b) Why is this called "modulation"? Explain from a communication systems perspective.(Optional)

(c) Given a rectangular pulse $x[n] = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$, find DTFT of $y[n] = x[n]\cos(\pi n/2)$.

Problem 4

Show that multiplication in time corresponds to periodic convolution in frequency.

1. Prove:

$$x[n]y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

Problem 5

Complete Derivation of Exponential Decay

(a) Derive DTFT of $x[n] = a^n u[n]$ for $|a| < 1$, showing all steps.

(b) Simplify to find explicit magnitude $|X(e^{j\omega})|$ and phase $\angle X(e^{j\omega})$.

Problem 6

LTI System Response via Frequency Domain (Cascade Analysis)

A system has impulse response $h[n] = (0.8)^n u[n]$ and receives input $x[n] = (0.5)^n u[n]$.

(a) Find $H(e^{j\omega})$ and $X(e^{j\omega})$ using standard pairs.

(b) Use the convolution property $Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$ to find $Y(e^{j\omega})$.

(c) Apply partial fraction decomposition to separate into known forms.

(d) Use inverse DTFT to find $y[n] = h[n] * x[n]$.

(e) Verify your answer by computing convolution directly in time domain for $n = 0, 1, 2$.

Problem 7

Parseval's Theorem: Energy Conservation in Transform

(a) Prove Parseval's theorem: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

(b) For $x[n] = (0.8)^n u[n]$, compute total energy both ways and verify equality.

Problem 8

Why DTFT Lacks Time-Scaling Property (Fundamental Limitation)

In continuous-time FT, the time-scaling property states: $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|}X(j\omega/a)$

(a) Explain why a direct time-scaling property does not exist for DTFT. What is the fundamental obstacle?

(b) What discrete-time operations (downsampling, upsampling) play analogous roles? How do they differ?

Problem 9

Find inverse DTFT:

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

— End of Problem Set —