

Key Concepts of the Discrete-Time Fourier Transform (DTFT)

The **Discrete-Time Fourier Transform (DTFT)** serves as the frequency-domain representation of a discrete-time signal. It allows us to decompose a sequence into its constituent complex exponential components.

The relationship between the time-domain sequence $x[n]$ and its frequency-domain representation $X(e^{j\omega})$ is defined by the following two equations:

- **Analysis Equation (Forward DTFT):** Converts the time-domain signal into the frequency domain.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Synthesis Equation (Inverse DTFT):** Reconstructs the time-domain signal from its frequency components.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

A critical distinction of the DTFT is its **periodicity**. Because $e^{-j(\omega+2\pi)n} = e^{-j\omega n} \cdot e^{-j2\pi n} = e^{-j\omega n}$ for any integer n , the spectrum repeats every 2π radians. Consequently, we typically analyze the DTFT over the fundamental interval $\omega \in [-\pi, \pi]$.

DTFT Properties

These properties allow us to manipulate signals in the time domain by performing simpler operations in the frequency domain.

Property	Time Domain $x[n]$	Frequency Domain $X(e^{j\omega})$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Convolution	$x[n] * h[n]$	$X(e^{j\omega})H(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's Relation	$\sum x[n] ^2$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) ^2 d\omega$

Common DTFT Pairs

The following pairs are frequently used to solve convolution and system response problems without evaluating the summation directly.

Signal Name	Time Domain $x[n]$	Frequency Domain $X(e^{j\omega})$
Unit Impulse	$\delta[n]$	1
Shifted Impulse	$\delta[n - n_0]$	$e^{-j\omega n_0}$
Exponential Decay	$a^n u[n], \quad a < 1$	$\frac{1}{1 - ae^{-j\omega}}$
Rectangular Pulse	$u[n] - u[n - N]$	$e^{-j\omega \frac{N-1}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$
Ideal Lowpass	$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \omega \leq \omega_c \\ 0, & \omega_c < \omega \leq \pi \end{cases}$
Constant (DC)	1 (for all n)	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

Convergence and Physical Interpretation

The DTFT converges if $x[n]$ is absolutely summable. The magnitude $|X(e^{j\omega})|$ describes the "strength" of various frequency components, while the phase $\angle X(e^{j\omega})$ captures the relative alignment (delay) of these frequencies. This preamble provides the theoretical foundation for the analytical problems presented in this set.

Problem 1

The DTFT is defined as: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

- (a) Prove that $X(e^{j\omega})$ is periodic with period 2π . Explain why this periodicity exists.
(b) Given $x[n] = \delta[n]$, compute the DTFT analytically and verify periodicity.

Solution:

Part (a): Periodicity Proof:

Substitute $(\omega + 2\pi)$ into the DTFT:

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \cdot e^{-j2\pi n}$$

Since $n \in \mathbb{Z}$, we have $e^{-j2\pi n} = \cos(2\pi n) - j \sin(2\pi n) = 1$ for all integer n .

Therefore: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Explanation: In continuous-time FT, frequencies are unbounded and non-periodic. In discrete-time, the exponential $e^{j\omega n}$ depends only on the fractional part of ωn . Since n is always an integer, adding 2π to ω produces no change in the exponential values, creating periodicity. This is fundamentally tied to discrete indexing.

Part (b): DTFT of $\delta[n]$:

Using the definition with $x[n] = \delta[n]$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0]e^0 = 1$$

Thus: $X(e^{j\omega}) = 1$ for all ω (constant for all frequencies)

Verification: $X(e^{j(\omega+2\pi)}) = 1 = X(e^{j\omega})$

Problem 2

Prove uniqueness: if $X_1(e^{j\omega}) = X_2(e^{j\omega})$ for all $\omega \in [-\pi, \pi]$, then $x_1[n] = x_2[n]$ for all n . What convergence conditions ensure the DTFT exists?

Solution: Assume $X_1(e^{j\omega}) = X_2(e^{j\omega})$ for all $\omega \in [-\pi, \pi]$.

Define $x_3[n] = x_1[n] - x_2[n]$.

Then $X_3(e^{j\omega}) = X_1(e^{j\omega}) - X_2(e^{j\omega}) = 0$ for all ω .

Using the inverse DTFT:

$$x_3[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_3(e^{j\omega})e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} 0 \cdot e^{j\omega n} d\omega = 0$$

Therefore $x_1[n] = x_2[n]$ for all n .

Implication: The DTFT transformation is one-to-one. Every signal has a unique frequency representation and vice versa. No information is lost.

Sufficient condition (Absolute Summability):

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \implies \text{DTFT converges pointwise}$$

This guarantees $|X(e^{j\omega})| \leq \sum |x[n]|$ (bounded).

Problem 3

Frequency-Shift (Modulation) Property

(a) Prove: if $x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega})$, then $e^{j\omega_0 n}x[n] \xrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)})$

(b) Why is this called “modulation”? Explain from a communication systems perspective. (Optional)

(c) Given a rectangular pulse $x[n] = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$, find DTFT of $y[n] = x[n] \cos(\pi n/2)$.

Solution: Part (a): Derivation

$$\begin{aligned} \mathcal{F}\{e^{j\omega_0 n} x[n]\} &= \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega - \omega_0)n} = X(e^{j(\omega - \omega_0)}) \quad \checkmark \end{aligned}$$

Part (b): Modulation in Communications(Optional)

In communication systems:

- **Baseband signal:** $x[n]$ occupies low frequencies (e.g., audio 20-20kHz)
- **Modulation:** Multiply by $e^{j\omega_0 n}$ shifts spectrum to carrier frequency ω_0
- **Purpose:** Antenna efficiently radiates at high frequencies (not low frequencies)
- **Reception:** Demodulate by multiplying received signal by $e^{-j\omega_0 n}$ to shift back
- **Bandpass filtering:** Select the desired modulated band from the shifted spectrum

This is the foundation of all modern communication: AM/FM radio, Wi-Fi, cellular, satellite.

Part (c): Rectangular Pulse \times Cosine

Step 1: Rectangular pulse DTFT:

$$x[n] \text{ (rect, length 5)} \xrightarrow{\mathcal{F}} X(e^{j\omega}) = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}} = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

Step 2: Express cosine as complex exponentials:

$$\cos(\pi n/2) = \frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2}$$

Step 3: Apply linearity and modulation:

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2} X(e^{j(\omega - \pi/2)}) + \frac{1}{2} X(e^{j(\omega + \pi/2)}) \\ &= \frac{1}{2} e^{-j2(\omega - \pi/2)} \frac{\sin(5(\omega - \pi/2)/2)}{\sin((\omega - \pi/2)/2)} + \frac{1}{2} e^{-j2(\omega + \pi/2)} \frac{\sin(5(\omega + \pi/2)/2)}{\sin((\omega + \pi/2)/2)} \end{aligned}$$

Interpretation: The rectangular pulse, when amplitude-modulated by a cosine at $\omega_0 = \pi/2$, creates two spectral lobes (AM double-sideband). This is fundamental to AM communication.

Problem 4

Show that multiplication in time corresponds to periodic convolution in frequency.

1. Prove:

$$x[n]y[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$$

Solution:

We want to prove:

$$x[n]y[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$$

Step 1: Start with inverse DTFT expressions

From inverse DTFT:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})e^{j\theta n}d\theta$$

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\phi}) e^{j\phi n} d\phi$$

Step 2: Multiply the signals

$$\begin{aligned} x[n]y[n] &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta n} d\theta \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\phi}) e^{j\phi n} d\phi \right) \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j\phi}) e^{j(\theta+\phi)n} d\theta d\phi \end{aligned}$$

Step 3: Take DTFT of the product

DTFT definition:

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-j\omega n}$$

Substitute expression:

$$\begin{aligned} Z(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j\phi}) e^{j(\theta+\phi)n} e^{-j\omega n} d\theta d\phi \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j\phi}) \left[\sum_{n=-\infty}^{\infty} e^{jn(\theta+\phi-\omega)} \right] d\theta d\phi \end{aligned}$$

Step 4: Use identity (periodic impulse train)

$$\sum_{n=-\infty}^{\infty} e^{jn\alpha} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\alpha - 2\pi k)$$

Apply with $\alpha = \theta + \phi - \omega$:

$$\sum_{n=-\infty}^{\infty} e^{jn(\theta+\phi-\omega)} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\theta + \phi - \omega - 2\pi k)$$

Step 5: Substitute back

$$\begin{aligned} Z(e^{j\omega}) &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j\phi}) \cdot 2\pi \sum_{k=-\infty}^{\infty} \delta(\theta + \phi - \omega - 2\pi k) d\theta d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j\phi}) \sum_{k=-\infty}^{\infty} \delta(\theta + \phi - \omega - 2\pi k) d\theta d\phi \end{aligned}$$

Step 6: Evaluate delta function

Using delta:

$$\phi = \omega - \theta + 2\pi k$$

Due to periodicity of DTFT:

$$Y(e^{j(\omega-\theta+2\pi k)}) = Y(e^{j(\omega-\theta)})$$

Thus integral reduces to:

$$Z(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$$

Final Result:

$$x[n]y[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$$

Key Insight:

- In continuous-time \rightarrow multiplication gives **convolution**
- In discrete-time \rightarrow multiplication gives **periodic convolution**
- Periodicity arises because DTFT is 2π -periodic

Problem 5

Complete Derivation of Exponential Decay

- (a) Derive DTFT of $x[n] = a^n u[n]$ for $|a| < 1$, showing all steps.
(b) Simplify to find explicit magnitude $|X(e^{j\omega})|$ and phase $\angle X(e^{j\omega})$.

Solution: Part (a): Derivation

By definition:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

This is a geometric series with ratio $r = ae^{-j\omega}$. Convergence: $|r| = |a| \cdot |e^{-j\omega}| = |a| < 1$

Using formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$:

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad \boxed{\text{Standard pair}}$$

Part (b): Magnitude and Phase

Multiply numerator and denominator by $e^{j\omega/2}$:

$$X(e^{j\omega}) = \frac{e^{j\omega/2}}{e^{j\omega/2} - ae^{-j\omega/2}}$$

Denominator: $e^{j\omega/2} - ae^{-j\omega/2} = (1-a)\cos(\omega/2) + j(1+a)\sin(\omega/2)$

Therefore:

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a)^2 \cos^2(\omega/2) + (1+a)^2 \sin^2(\omega/2)}}$$

Expanding:

$$= \frac{1}{\sqrt{1 + a^2 - 2a \cos \omega}} \quad \checkmark$$

Phase:

$$\angle X(e^{j\omega}) = -\arctan\left(\frac{a \sin \omega}{1 - a \cos \omega}\right)$$

Problem 6

LTI System Response via Frequency Domain (Cascade Analysis)

A system has impulse response $h[n] = (0.8)^n u[n]$ and receives input $x[n] = (0.5)^n u[n]$.

- (a) Find $H(e^{j\omega})$ and $X(e^{j\omega})$ using standard pairs.
(b) Use the convolution property $Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$ to find $Y(e^{j\omega})$.
(c) Apply partial fraction decomposition to separate into known forms.
(d) Use inverse DTFT to find $y[n] = h[n] * x[n]$.
(e) Verify your answer by computing convolution directly in time domain for $n = 0, 1, 2$.

Solution: Part (a): Frequency Responses

Using standard pairs:

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1 - 0.8e^{-j\omega}}$$

Part (b): Product in Frequency Domain

By convolution property:

$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega}) = \frac{1}{(1 - 0.8e^{-j\omega})(1 - 0.5e^{-j\omega})}$$

Part (c): Partial Fraction Decomposition

Let:

$$\frac{1}{(1 - 0.8e^{-j\omega})(1 - 0.5e^{-j\omega})} = \frac{A}{1 - 0.8e^{-j\omega}} + \frac{B}{1 - 0.5e^{-j\omega}}$$

Multiply both sides by the denominator:

$$1 = A(1 - 0.5e^{-j\omega}) + B(1 - 0.8e^{-j\omega})$$

Setting $e^{-j\omega} = 1/0.8$ (pole of first term):

$$1 = A(1 - 0.5/0.8) = A(0.375) \implies A = \frac{1}{0.375} = \frac{8}{3}$$

Setting $e^{-j\omega} = 1/0.5$ (pole of second term):

$$1 = B(1 - 0.8/0.5) = B(-0.6) \implies B = -\frac{5}{3}$$

Therefore:

$$Y(e^{j\omega}) = \frac{8/3}{1 - 0.8e^{-j\omega}} - \frac{5/3}{1 - 0.5e^{-j\omega}}$$

Part (d): Inverse DTFT

Using the standard pair $a^n u[n] \xrightarrow{\mathcal{F}} \frac{1}{1 - ae^{-j\omega}}$:

$$y[n] = \left(\frac{8}{3}(0.8)^n - \frac{5}{3}(0.5)^n \right) u[n]$$

Part (e): Time-Domain Verification

By definition: $y[n] = \sum_{k=0}^n x[k]h[n-k]$

For $n = 0$: $y[0] = x[0]h[0] = 1 \cdot 1 = 1$

Check formula: $\frac{8}{3}(0.8)^0 - \frac{5}{3}(0.5)^0 = \frac{8}{3} - \frac{5}{3} = 1$

For $n = 1$:

$$y[1] = x[0]h[1] + x[1]h[0] = 1 \cdot 0.8 + 0.5 \cdot 1 = 1.3$$

Check formula: $\frac{8}{3}(0.8) - \frac{5}{3}(0.5) = \frac{6.4}{3} - \frac{2.5}{3} = \frac{3.9}{3} = 1.3$

For $n = 2$:

$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] = 1(0.64) + 0.5(0.8) + 0.25(1) = 1.29$$

Check formula: $\frac{8}{3}(0.64) - \frac{5}{3}(0.25) = \frac{5.12}{3} - \frac{1.25}{3} = \frac{3.87}{3} \approx 1.29$

Problem 7

Parseval's Theorem: Energy Conservation in Transform

(a) Prove Parseval's theorem: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

(b) For $x[n] = (0.8)^n u[n]$, compute total energy both ways and verify equality.

Solution: Part (a): Proof

Start with the inverse DTFT:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Multiply both sides by $x^*[n]$ and sum over all n :

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) e^{-j\omega n} d\omega$$

Interchange sum and integral (valid for convergent signals):

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) \left[\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right] d\omega$$

The sum in brackets is $X(e^{j\omega})$ by DTFT definition:

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega})X(e^{j\omega})d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \quad \checkmark$$

Significance: The theorem states that energy is conserved in the Fourier transform. You can compute total signal energy either in time domain or frequency domain—the answer is always the same. This validates that frequency-domain analysis is physically meaningful.

Part (b): Numerical Verification

For $x[n] = (0.8)^n u[n]$:

Time-domain energy:

$$E = \sum_{n=0}^{\infty} |(0.8)^n|^2 = \sum_{n=0}^{\infty} (0.64)^n = \frac{1}{1-0.64} = \frac{1}{0.36} \approx 2.778$$

Frequency-domain energy:

From earlier: $X(e^{j\omega}) = \frac{1}{1-0.8e^{-j\omega}}$

$$|X(e^{j\omega})|^2 = \frac{1}{1+0.64-1.6\cos\omega} = \frac{1}{1.64-1.6\cos\omega}$$

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1.64-1.6\cos\omega} d\omega$$

Using the standard integral formula $\int_0^{2\pi} \frac{d\theta}{a-b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ for $a > b$:

$$\int_{-\pi}^{\pi} \frac{d\omega}{1.64-1.6\cos\omega} = \frac{2\pi}{\sqrt{1.64^2-1.6^2}} = \frac{2\pi}{\sqrt{2.6896-2.56}} = \frac{2\pi}{\sqrt{0.1296}} = \frac{2\pi}{0.36}$$

$$E = \frac{1}{2\pi} \cdot \frac{2\pi}{0.36} = \frac{1}{0.36} \approx 2.778 \quad \checkmark$$

Perfect match! Parseval's theorem verified.

Problem 8

Why DTFT Lacks Time-Scaling Property (Fundamental Limitation)

In continuous-time FT, the time-scaling property states: $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X(j\omega/a)$

- (a) Explain why a direct time-scaling property does not exist for DTFT. What is the fundamental obstacle?
- (b) What discrete-time operations (downsampling, upsampling) play analogous roles? How do they differ?

Solution: Part (a): Why Time-Scaling Fails in Discrete-Time

In continuous time, scaling by a means: input at time at is replaced with $x(at)$. The function is defined everywhere, and scaling is well-defined.

In discrete time, signals are defined only at integer indices $n \in \mathbb{Z}$. What does $x(an)$ mean if a is non-integer?

If $a = 2$ (downsampling):

$$x[2n] = \{x[0], x[2], x[4], x[6], \dots\}$$

This is well-defined but represents **sample skipping**, losing information. Not a simple scaling.

If $a = 0.5$ (upsampling):

$$x[0.5n] \text{ at integer } n : \quad x[0], x[0.5], x[1], x[1.5], \dots$$

Values at $n = 0.5, 1.5, \dots$ are **undefined!** Would require interpolation (creating new samples).

If $a = \sqrt{2}$ (irrational):

$$x[\sqrt{2}n] = x[0], x[\sqrt{2}], x[2\sqrt{2}], x[3\sqrt{2}], \dots$$

Mostly undefined at integer indices. Cannot be evaluated.

Conclusion: The fundamental obstacle is discrete indexing. Scaling a continuous domain doesn't preserve the integer-index property, so a direct time-scaling transform property is impossible. The DTFT's periodicity also prevents the simple frequency scaling seen in continuous FT.

Part (b): Discrete Analogs—Multirate Operations

Downsampling by M :

$$y[n] = x[Mn]$$

Decimation: Keep every M -th sample, discard others. Spectrum stretches and aliases.

Upsampling by L :

$$z[n] = \begin{cases} x[n/L] & n = 0, L, 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$

Insertion of zeros: Compress spectrum by creating $L - 1$ zero values between original samples. Spectrum shrinks and images appear.

Unlike scaling: Both operations involve frequency distortion (aliasing/imaging) that scaling avoids. Thus they are not true "scalings" but rather resampling operations.

Problem 9

Find inverse DTFT:

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

Solution:

Step 1: Use IDTFT definition

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Since $X = 1$ only in given region:

$$x[n] = \frac{1}{2\pi} \left[\int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega + \int_{-3\pi/4}^{-\pi/4} e^{j\omega n} d\omega \right]$$

Step 2: Evaluate integrals

$$\int e^{j\omega n} d\omega = \frac{e^{j\omega n}}{jn}$$

First integral:

$$\frac{e^{j3\pi n/4} - e^{j\pi n/4}}{jn}$$

Second:

$$\frac{e^{-j\pi n/4} - e^{-j3\pi n/4}}{jn}$$

Step 3: Combine

$$x[n] = \frac{1}{2\pi jn} \left[e^{j3\pi n/4} - e^{j\pi n/4} + e^{-j\pi n/4} - e^{-j3\pi n/4} \right]$$

Step 4: Use identity

$$e^{j\theta} - e^{-j\theta} = 2j \sin \theta$$

$$x[n] = \frac{1}{\pi n} \left[\sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{\pi n}{4}\right) \right]$$

$$x[n] = \frac{1}{\pi n} \left[\sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{\pi n}{4}\right) \right]$$

— End of Problem Set —