LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE563 Convex Optimization
Spring 2020

Assignment-1

Posted/Updated on: February 09, 2020
Submission Deadline: Wednesday, February 12, 2020
NOTE: Hand in your assignment in class or drop-off your assignment in the dropbox located outside 9-251 (EE Right wing).

Problem 1 (10 points)
Consider the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 & 1 \\ 2 & 6 & 3 & 11 & 1 \\ 1 & 4 & 2 & 7 & 0 \end{bmatrix}$$

a) Find the basis vectors of its null space.

b) Find its singular values.

c) Is $A^T A$ positive definite? Explain.

Problem 2 (6 points)
Reason out the fact that $x = (A^T A)^{-1} A^T b$ is the solution to the least-squares optimization problem defined in section 1.2.1 of the textbook using the concept of null and column spaces of matrix A.

Problem 3 (14 points)
Find out whether the following sets are convex or not.

a) Set $S = \{a \in \mathbf{R}^k \mid p(0) = 1, \; |p(t)| \leq 1 \; \forall \; \alpha \leq t \leq \beta\}$, where

$$p(t) = a_1 + a_2 t + ... + a_k t^{k-1},$$

b) An interval $[a, b] \subset \mathbf{R}$

c) Set $S = \{x \in \mathbf{R}^2 \mid a^T x \leq c\}, \quad a \in \mathbf{R}^n, c \in \mathbf{R}$

d) Set $S = \{x \in \mathbf{R}^n \mid Ax = b\}, \quad b \in \mathbf{R}^m$ and A is a matrix with dimensions m x n

e) Set $S = \{x \in \mathbf{R}^2 \mid e^{x_1} < x_2\}$

f) Set of points on the surface of a unit n-sphere

Problem 4 (8 points)
Describe the dual cone for each of the following cones.

a) $K = \{0\}$

b) $K = \mathbf{R}^2$

c) $K = \{(x_1, x_2) \mid |x_1| < x_2\}$

d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$

Problem 5 (8 points)
The polar of $C \subseteq \mathbf{R^n}$ is defined as the set

$$C^o = \{y \in \mathbf{R^n} \mid y^T x \leq 1 \;\forall\; x \in C\}.$$

a) Show that C^o is convex (even if C is not).

b) What is the polar of a cone?

c) What is the polar of the unit ball for a norm $||\,.\,||$?

d) Show that if C is closed and convex, with $0 \in \mathbf{int}C$, then $(C^o)^o =$ C.

Problem 6 (6 points)
Show that if $C \in \mathbf{R}^n$ is convex, then closure of C is also convex.

Problem 7 (6 points)
Prove that the following are convex.

a) An ellipsoid

b) A norm cone

Problem 8 (8 points) For a symmetric matrix $A \in S^n$ with eigenvalues denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, show that

a) the eigenvalues are real.

b)
$$tr(A^k) = \sum_{i=1}^{n} \lambda_i^k.$$
where k denotes any positive integer.

Problem 9 (6 points)

Given the convex sets C_1 and C_2 in $\mathbf{R}^{m+n}$, show that the partial sum given by

$$C = \{(x, y_1 + y_2) | x \in \mathbf{R}^m,\ y_1, y_2 \in \mathbf{R}^n,\ x \in C_1,\ y_1, y_2 \in C_2\}$$

is a convex set.

Problem 10 (8 points)
A matrix $X \in \mathbf{S^n}$ is called copositive if $z^T X z \geq 0 \ \forall \ z \succeq 0$. Verify that the set of copositive matrices is a proper cone. Also find its dual cone.

Problem 11 (10 points) For a linear-fractional function $f : \mathbf{R}^m \to \mathbf{R}^n$ given by

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \mathbf{dom} f = \{x | c^T x + d > 0\},$$

the inverse image of a convex set under f is defined as

$$f^{-1}(C) = \{x \in \mathbf{dom} f | f(x) \in C\}.$$

For each of the following sets, give a simple description of the set and its inverse $f^{-1}(C)$.

a) $C = \{y | \, g^T y \leq h, g \neq 0\}$

b) $C = \{y | \, Gy \preceq h, g \neq 0\}$

c) $C = \{y | \, y^T P^{-1} y \leq 1, \ P \in \mathbf{S}^n_{++}\}$