LUMS
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EE563 Convex Optimization

Assignment 01 Solution

Problem 1

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 & 1 \\ 2 & 6 & 3 & 11 & 1 \\ 1 & 4 & 2 & 7 & 0 \end{bmatrix}_{3\times 5}$$

a) Convert the matrix to reduced form by applying row operations:

$$A \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 4 & 1 \\ 0 & 2 & 1 & 3 & -1 \\ 0 & 2 & 1 & 3 & -1 \end{bmatrix} \xrightarrow[R_3 - 2R_2]{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 1 & 4 & 1 \\ 0 & 1 & 1/2 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow R_1 - 2R_2$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1/2 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Now, $Ax = 0 \Rightarrow Rx = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1/2 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ a & b are pivot while c, d & e are free variables.

$$a + d + 2e = 0 \quad \Rightarrow \quad a = -d - 2e$$

$$b + \frac{c}{2} + \frac{3d}{2} - \frac{e}{2} = 0 \quad \Rightarrow \quad b = -\frac{c}{2} - \frac{3d}{2} + \frac{e}{2}$$

$$\Rightarrow x = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -d-2e \\ -\frac{c}{2} - \frac{3}{2}d + \frac{e}{2} \\ c \\ d \\ e \end{bmatrix} = c\begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ -3/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e\begin{bmatrix} -2 \\ 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis vectors of Nullspace of A.

b) Singular Values are the square root of eigenvalues of A^TA & AA^T.

$$AA^T = \begin{bmatrix} 23 & 62 & 39 \\ 62 & 171 & 109 \\ 39 & 109 & 70 \end{bmatrix}$$

Its eigenvalues can be found to be $\lambda = 0, 1.02, 263$

$\Rightarrow$ Singular values of A are $\sigma = 0, 1.01, 16.22$

c) Eigenvalues of $A^T A$ are the same as that of $A A^T$,

$\Rightarrow$ $A^T A$ is +ve semi-definite & not +ve definite because it has 3 zero eigenvalues. ($A^T A$ will have 5 eigenvalues instead of 3 so atleast two of them will be zero)

Problem 2

System of equations represented by $Ax = b$ has a definite solution, $x = A^{-1}b$, if 'b' lies in the column space of 'A'. If not, then the closest match to 'b' is its projection on the column space (which minimizes the error in least-squares sense).

Let $p = A\hat{x}$ be the projection of 'b' onto $C(A)$. Then, $(b - A\hat{x})$ is the component of 'b' $\perp$ to $C(A)$ $\Rightarrow$ $(b - A\hat{x})$ lies in the left Null space of A. Hence,

$$A^T(b - A\hat{x}) = 0 \quad \text{or} \quad A^T A\hat{x} = A^T b$$

$$\Rightarrow \quad \hat{x} = (A^T A)^{-1} A^T b$$

Problem 3

a) $S = \{a \in \mathbb{R}^k \mid p(0) = 1,\ |p(t)| \leq 1 \ \forall\ \alpha \leq t \leq \beta\}$, where

$$p(t) = a_1 + a_2 t + \dots + a_k t^{k-1}$$

S is convex. $\quad p(t) = a^T t \quad$ where $\quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$ & $t = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{k-1} \end{bmatrix}$

$$|p(t)| \leq 1 \quad \Rightarrow \quad -1 \leq p(t) \leq 1$$

which defines the intersection of two half planes. (Half plane is a convex set). Now, the constraint $p(0) = 1 \Rightarrow a_1 = 1$ which is a line in k-dimensional space.

$\Rightarrow$ S is an intersection of two half planes & a line, all of which are convex sets. Since, intersection of convex sets is convex, S is convex.

b) An interval $[a, b] \subset \mathbb{R}$

Since, $[a, b]$ is a closed interval on real line, line segment b/w any two points in the interval is alway $\subseteq [a, b] \subset \mathbb{R}$, which makes it convex by definition.

c) $S = \{x \in \mathbb{R}^2 \mid a^T x \leq c\}$, $a \in \mathbb{R}^n$, $c \in \mathbb{R}$

S is a half space in $\mathbb{R}^2$ & hence, is convex.

d) $S = \{x \in \mathbb{R}^n \mid Ax = b\}$, $b \in \mathbb{R}^m$ & $A \in \mathbb{R}^{m \times n}$

Let $y, z \in S$ & $0 \leq \theta \leq 1$. Then

$$\theta y + (1-\theta) z = \theta (A^{-1} b) + (1-\theta) A^{-1} b = A^{-1} b = x \in S$$

Hence, S is convex.

e) $S = \{x \in \mathbb{R}^2 \mid e^{x_1} < x_2\}$

OR $S = \{x \in \mathbb{R}^2 \mid x_2 > e^{x_1}\}$

Let $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ & $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in S$ i.e.,

$y_2 > e^{y_1}$ & $z_2 > e^{z_1}$, then for $0 \leq \theta \leq 1$;

$\theta y_2 > \theta e^{y_1}$ & $(1-\theta) z_2 > e^{z_1}(1-\theta)$ $(\because 0 \leq 1-\theta \leq 1)$

$\Rightarrow \theta y_2 + (1-\theta) z_2 > \theta e^{y_1} + (1-\theta) e^{z_1} \Rightarrow \theta y + (1-\theta) z \in S.$

Hence, S is Convex.

f) $S = \{x \in \mathbb{R}^n \mid \|x - x_c\|_2 = 1\}$ where x_c is the centre of the sphere.

Let y & z be two points on the surface of the sphere. Then,

$\|y - x_c\|_2 = 1$ & $\|z - x_c\|_2 = 1$. Let $0 \leq \theta \leq 1$. Then,

$$\begin{aligned} \|\theta y + (1-\theta) z - x_c\|_2 \quad & \|\theta (y - x_c) + (1-\theta)(z - x_c)\|_2 \\ & \leq \theta \|y - x_c\|_2 + (1-\theta) \|z - x_c\|_2 \\ & \leq \theta + 1 - \theta \\ & \leq 1 \end{aligned}$$

$\Rightarrow$ S is non-convex.

Problem 4

a) $K = \{0\}$

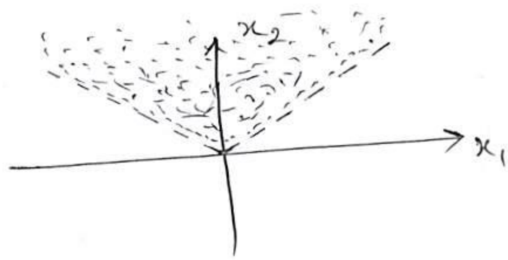
Its dual cone is the whole space.

b) $K = \mathbb{R}^2$

Since $\{0\}$ is a proper cone & dual of the dual of a cone is the cone itself, dual cone of $K = \mathbb{R}^2$ is $K = \{0\}$.

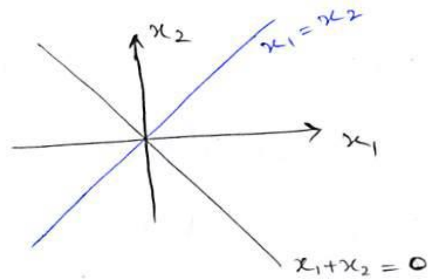
c) $K = \{(x_1, x_2) \mid |x_1| < x_2\}$

Dual cone of K is K itself + its boundary.



d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$

Its dual cone is $K = \{(x_1, x_2) \mid x_1 - x_2 = 0\}$



Problem 5

$$C^\circ = \{ y \in \mathbb{R}^n \mid y^T x \le 1 \ \forall \ x \in C\} \ ; \ C \subseteq \mathbb{R}^n$$

a) Let $w, z \in C^\circ$ & $y \in C$ such that:

$$z^T y \le 1 \quad \& \quad w^T y \le 1$$

Then for $0 \le \theta \le 1$;

$$[\theta w + (1-\theta) z]^T y = \theta w^T y + (1-\theta) z^T y \le \theta + 1 - \theta \le 1$$

$\Rightarrow \theta w + (1-\theta) z \in C^\circ$ & hence, C° is convex.

b) Since any cone K is closed under non-negative scaling, the polar of a cone K can be equivalently defined as

$$K^\circ = \{ y \in \mathbb{R}^n \mid y^T x \le 1 \ \forall x \in K \}$$
$$= \{ y \in \mathbb{R}^n \mid y^T x \le 0, \ \forall x \in K \}$$

and is therefore

$$K^\circ = \mathbb{R}^n - K^* \equiv -K^* \quad (\text{Dual cone } K^*)$$

c) Let B_p denote Ball of p-norm

Polar: $B_p^{\circ} = \{ y \in \mathbb{R}^n \mid y^T x \le 1, \ \underbrace{\|x\|_p \le 1}_{\text{unit p-ball}} \}$.

$= \{ y \in \mathbb{R}^n \mid \sup_{\|x\|_p \le 1} y^T x \le 1 \}$.

$= \{ y \in \mathbb{R}^n \mid \|y\|_q \le 1 \}$.

$\| \ \|_q$ is the dual norm associated with p-norm such that $\frac{1}{p} + \frac{1}{q} = 1$

Special cases: $B_1^{\circ} = B_{\infty}$, $B_{\infty}^{\circ} = B_1$

d) Previous parts are special cases ~~of~~ that ~~a~~ polar of a polar of a set (convex, closed) is set itself

For convex set C, $C^{\circ} = \{ y \mid y^T x \le 1 \ \forall x \in C \}$. $(C^{\circ})^{\circ} = \{ z \mid z^T y \le 1 \ \forall z \in C^{\circ} \}$

For all $y \in C^{\circ}$, we have $z^T y \le 1$. OR $y \in (C^{\circ})^{\circ}$, which implies.
$C \subseteq (C^{\circ})^{\circ}$. If C is closed, $C = (C^{\circ})^{\circ}$.

Problem 6 Suppose x & $y \in cl(C)$ where 'cl' denotes closure of a set. Now there exists sequences $\{x_n\}_1^{\infty}$ & $\{y_n\}_1^{\infty}$ in C such that $x_n \to x$ & $y_n \to y$. For $0 \le \theta \le 1$, let $z_n = \theta x_n + (1-\theta) y_n$ which belongs to C due to its convexity. Then, $z_n \longrightarrow z$ which implies $z_n \in cl(C)$. Hence, closure of C is convex.

Problem 7

a) $\mathcal{E} = \{x \in \mathbb{R}^n \mid (x-x_c)^T P^{-1}(x-x_c) \leq 1\}$, where $P = P^T \succ 0$ & x_c is the centre of ellipsoid.

Since, $P = P^T$, $P^{-1} = (P^T)^{-1} = (P^{-1})^T \succ 0$.

Let $P^{-1} = Q$, $\Rightarrow$ $Q = Q^T \succ 0$.

Now a real symmetric matrix (Q) is +ve definite iff:

$$Q = M^T M \quad , \text{ where } M \text{ is non-singular real matrix.}$$

$\Rightarrow$ $(x-x_c)^T M^T M(x-x_c) = \|(x-x_c)M\|_2^2$. Hence, ellipsoid can be redefined as: $\mathcal{E} = \{x \in \mathbb{R}^n \mid \|(x-x_c)M\|_2 \leq 1\}$

Let $y, z \in \mathcal{E}$ such that $\|(y-x_c)M\|_2 \leq 1$ & $\|(z-x_c)M\|_2 \leq 1$.

Then,
$$\begin{aligned}
\|(\theta y + (1-\theta)z)M\|_2 &= \|(\theta(y-x_c) + (1-\theta)(z-x_c))M\|_2 \\
&\leq \theta\|(y-x_c)M\|_2 + (1-\theta)\|(z-x_c)M\|_2 \\
&\leq \theta + 1 - \theta \\
&\leq 1 \quad \text{for } 0 \leq \theta \leq 1.
\end{aligned}$$

$\Rightarrow$ $\mathcal{E}$ is Convex.

b) $C = \{(x,t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$ where $\|.\|$ is any norm on $\mathbb{R}^n$.

Let $y = (x_1, t_1)$ & $z = (x_2, t_2) \in C$ & $0 \leq \theta \leq 1$. Then,

$$\begin{aligned}
\|\theta x_1 + (1-\theta)x_2\| &\leq \theta\|x_1\| + (1-\theta)\|x_2\| \\
&\leq \theta t_1 + (1-\theta)t_2 \quad \text{—— ①}
\end{aligned}$$

$\theta x_1 + (1-\theta)x_2$ is the x-part of convex combination of y & z.

Similarly $\theta t_1 + (1-\theta)t_2$ is the t-part of convex combination of y & z.

① $\Rightarrow$ $\theta y + (1-\theta)z \in C$ & hence, C is convex.

Problem 8

a) Characteristic polynomial, in general, may have complex roots. Contrary to the statement, assume that λ (root) is a complex number, and the associated eigenvector x may be complex-valued. By definition, we have $\boxed{Ax = \lambda x}$

By taking conjugate, denoted by $\overline{(\cdot)}$, we get

$$A\bar{x} = \bar{\lambda}\bar{x} \Rightarrow x^T A \bar{x} = \bar{\lambda} x^T \bar{x} \quad -①$$

$$Ax = \lambda x \Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x \quad -②$$

subtracting yields $\bar{x}^T A x - x^T A \bar{x} = (\lambda - \bar{\lambda}) x^T \bar{x}$

Since A is symmetric, the left hand side is zero, that is, $\lambda - \bar{\lambda} = 0 \Rightarrow$ $\underline{\lambda \text{ is real.}}$

b) We know we can decompose $A \in S^n$ as

$$A = Q \Lambda Q^T,$$

where Q is orthogonal matrix, i.e., $Q^T Q = I$

$$A^K = Q \Lambda^K Q^T, \qquad tr(A^K) = \sum_{i=1}^{n} q_i^T q_i \lambda_i^K = \sum_{i=1}^{n} \lambda_i^K,$$

where q_i is the i-th row of Q.

Problem 9

$$C = \{ (x, y_1 + y_2) \mid x \in R^m,\ y_1, y_2 \in R^n,\ (x, y_1) \in C_1, (x, y_2) \in C_2 \}.$$

Consider two points $(x_1, y_1 + y_2) \in C$

$(x_2, y_3 + y_4) \in C$

$\Rightarrow$

$(x_1, y_1) \in C_1 \qquad (x_1, y_2) \in C_2$

$(x_2, y_3) \in C_1 \qquad (x_2, y_4) \in C_2$

Following definition of convexity;

$$\theta (x_1, y_1 + y_2) + (1-\theta)(x_2, y_3 + y_4)$$

$$= (\theta x_1 + (1-\theta) x_2,\ \theta y_1 + (1-\theta) y_3 + \theta y_2 + (1-\theta) y_4) \in C$$

since we have

$(\theta x_1 + (1-\theta) x_2,\ \theta y_1 + (1-\theta) y_3) \in C_1$

$(\theta x_1 + (1-\theta) x_2,\ \theta y_2 + (1-\theta) y_4) \in C_2$

convex sets.

Problem 10

$z^T X z \geq 0$ is a system of homogenous inequalities.

Since, each inequality is a halfspace in $\mathbb{R}^n$, $z^T X z \geq 0$ represents the intersection of these halfspaces $\Rightarrow$ K is a closed convex cone. K has a non-empty interior because it includes the cone of +ve semidefinite matrices whose interior is non-empty.

K is pointed because for any $X \in K$, $-X \in K \Rightarrow z^T X z = 0$ $\forall\ z \succeq 0 \Rightarrow X = 0$.

Dual cone of K includes the normal vectors of all halfspaces of K & the origin as well,

$$K^* = \mathrm{conv}\{ z z^T \mid z \succeq 0 \}$$

Problem 11

(a) The halfspace $C = \{y \mid g^T y \leq h\}$ (with $g \neq 0$).
Solution.

$$\begin{aligned} f^{-1}(C) &= \{x \in \mathbf{dom}\, f \mid g^T f(x) \leq h\} \\ &= \{x \mid g^T (Ax+b)/(c^T x + d) \leq h,\ c^T x + d > 0\} \\ &= \{x \mid (A^T g - hc)^T x \leq hd - g^T b,\ c^T x + d > 0\}, \end{aligned}$$

which is another halfspace, intersected with $\mathbf{dom}\, f$.

(b) The polyhedron $C = \{y \mid Gy \preceq h\}$.
Solution. The polyhedron

$$\begin{aligned} f^{-1}(C) &= \{x \in \mathbf{dom}\, f \mid Gf(x) \preceq h\} \\ &= \{x \mid G(Ax+b)/(c^T x + d) \preceq h,\ c^T x + d > 0\} \\ &= \{x \mid (GA - hc^T)x \preceq hd - Gb,\ c^T x + d > 0\}, \end{aligned}$$

a polyhedron intersected with $\mathbf{dom}\, f$.

(c) The ellipsoid $\{y \mid y^T P^{-1} y \leq 1\}$ (where $P \in \mathbf{S}^n_{++}$).
Solution.

$$\begin{aligned} f^{-1}(C) &= \{x \in \mathbf{dom}\, f \mid f(x)^T P^{-1} f(x) \leq 1\} \\ &= \{x \in \mathbf{dom}\, f \mid (Ax+b)^T P^{-1}(Ax+b) \leq (c^T x + d)^2\}, \\ &= \{x \mid x^T Q x + 2q^T x \leq r,\ c^T x + d > 0\}. \end{aligned}$$

where $Q = A^T P^{-1} A - cc^T$, $q = b^T P^{-1} A + dc$, $r = d^2 - b^T P^{-1} b$. If $A^T P^{-1} A \succ cc^T$ this is an ellipsoid intersected with $\mathbf{dom}\, f$.