LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE563 Convex Optimization
Spring 2020

Assignment-2

Posted/Updated on: Wednesday, February 19, 2020
Submission Deadline: 6 pm, Monday, March 02, 2020
NOTE: Hand in your assignment in class or drop-off your assignment at Instructor's office (9-251, EE Right wing).

Problem 1) (20 points) Determine whether the following functions are convex, quasi-convex, concave, quasi-concave. Provide justification to support your answer.

a) $f(x) = \sqrt{x}\,, \quad \mathbf{dom} f = \mathbf{R}_+$

b) $f(x_1, x_2) = \frac{1}{x_1 x_2}\,, \quad \mathbf{dom} f = \mathbf{R}_{++}$

c) $f(x, Q) = x^T Q x\,, \quad \mathbf{dom} f = \mathbf{R}_{++} \times \mathbf{S}^n_{++}$

d) $f(x) = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad 0 \le p \le 1$

e) $f(x, y, z) = \ln(xyz)\,, \mathbf{dom} f = \{(x, y, z) \,|\, x > 0, y > 0, z > 0\}$

f) $\frac{1}{x^p}\,, \quad p > 0, \mathbf{dom} f = \mathbf{R}_{++}$

g) $f(x) = -\sqrt[n]{\prod_{i=1}^{n} x_i}\,, \quad \mathbf{dom} f = \mathbf{R}^n_{++}$

h) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}\,, \quad 0 \le \alpha \le 1\,, \mathbf{dom} f = \mathbf{R}^n_{++}$

Problem 2) (5 points) Consider a composite function $f = g \circ U$, given by

$$f(x) = g(U(x))$$

where $U : \mathbf{R}^k \to \mathbf{R}$ is a nondecreasing function and $g : \mathbf{R} \to \mathbf{R}$. If U and g are concave, show that f is concave.

Problem 3) (8 points)
The scalar Huber function is defined as:

$$f_{hub}(x) = \begin{cases} \frac{1}{2}x^2 & |x| \le 1 \\ |x| - \frac{1}{2} & |x| > 1 \end{cases}$$

This convex function comes up in several applications, including robust estimation. This problem concerns generalizations of the Huber function to $\mathbf{R}^n$. One generalization to $\mathbf{R}^n$ is given by $f_{hub}(x_1) + \ldots + f_{hub}(x_n)$, but this function is not circularly symmetric, i.e., invariant under transformation of x by orthogonal matrix. A generalization to $\mathbf{R}^n$ that is circularly symmetric is

$$f_{cshub}(x) = f_{hub}(||x||) = \begin{cases} \frac{1}{2}||x||_2^2 & ||x||_2 \leq 1 \\ ||x||_2 - \frac{1}{2} & ||x||_2 > 1. \end{cases}$$

(The subscript stands for 'circularly symmetric Huber function'). Show that f_{cshub} is convex.

Problem 4 (6+6+8 points)
A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be symmetric if it is invariant with respect to a permutation of its arguments, i.e., $f(x) = f(Px)$ for any permutation matrix P. In this problem we show that if f is convex and symmetric, then the function $g : \mathbf{S}^n \to \mathbf{R}$ defined as $g(X) = f(\lambda(X))$ is convex, where $\lambda(X) = (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))$ is the vector of eigenvalues of X. This implies, for example, that the function

$$g(X) = \log \mathbf{tr} e^X = \log \sum_{k=1}^{n} e^{\lambda_k(X)}$$

is convex on $\mathbf{S}^n$.

a) A square matrix S is *doubly stochastic* if its elements are nonnegative and all row sums and column sums are equal to one. It can be shown that every doubly stochastic matrix is a convex combination of permutation matrices.
Show that if f is convex and symmetric and S is doubly stochastic, then

$$f(Sx) \leq f(x)$$

b) Let $Y = Q\mathbf{diag}(\lambda)Q^T$ be an eigenvalue decomposition of $Y \in \mathbf{S}^n$ with Q orthogonal. Show that $n \times n$ matrix S with elements $S_{ij} = Q_{ij}^2$ is doubly stochastic and that $\mathbf{diag}(Y) = S\lambda$.

c) Use the results in parts (a) and (b) to show that if f is convex and symmetric and $X \in \mathbf{S}^n$, then
$$f(\lambda(X)) = \sup_{V \in \nu} f(\mathbf{diag}(V^T X V))$$
where ν is the set of $n \times n$ orthogonal matrices. Show that this implies that $f(\lambda(X))$ is convex in X.

Problem 5 (8 points) Consider a function $f(x,t) = -log(t^2 - x^T x)$, with $\mathbf{dom}\, f = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} |\ t > \|x\|_2\}$. Assuming that the function $x^T x/t$ (quadratic over linear) is convex on $\mathbf{dom}\, f$, use composition rules to establish the convexity of the the function f.

Problem 6 (6 points) Consider a linear fractional function $f : \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x) = \frac{a^T x + b}{c^T x + d}$$

with **dom** $f = \{x \in \mathbf{R}^n | \ c^T x + d > 0\}$. Show that the linear fractional function is quasiconvex. Under what conditions, the function f is convex?

Problem 7 (8 points) Determine the conjugate function of log-sum-exponential function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ given by

$$f(x) = \log \sum_{i=1}^{n} e^{x_i}$$

with **dom** $f = \mathbf{R}^n$.

Problem 8 (12 points) Determine whether the following functions are log-concave or log-convex.

a) Gamma function $\Gamma : \mathbf{R} \rightarrow \mathbf{R}$ given by:

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

.

b) $f(X) = \frac{\det(X)}{\text{trace}(X)}$, **dom** $f = \mathbf{S}^n_{++}$

c) $f(x) = \frac{e^x}{1+e^x}$.

Problem 9 (13 points) The Minkowski function of a convex set C is defined as

$$M_C(x) = inf\{t > 0 \ | \ t^{-1}x \in C\}.$$

a) Draw a picture giving a geometric interpretation of how to find $M_C(x)$.

b) Show that $M_C(x)$ is homogeneous, i.e., $M_C(\alpha x) = \alpha M_C(x) \forall \alpha \geq 0$.

c) What is **dom**M_C?

d) Show that M_C is a convex function.

e) Suppose C is also closed, bounded, symmetric (if $x \in C$ then $-x \in C$), and has non-empty interior. Show that M_C is a norm. What is the corresponding unit ball?