LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE563 Convex Optimization
Spring 2020

Assignment-3

Posted on: Monday, March 24, 2020
Submission Deadline: 6 pm, Friday, April 17, 2020
NOTE: Submit the assignment through LMS.

Problem 1) (10 points)
Formulate the following optimization problems as linear programs. Here $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$, and $b \in \mathbf{R}^m$.

a) minimize $\|Ax - b\|_\infty$

b) minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$

c) minimize $\|Ax - b\|_1 + \|x\|_\infty$

Problem 2) (10 points)
Optimal Control of a Linear Dynamical System
System Model: Consider a linear system with input $u(t) \in \mathbf{R}$, where t denotes the time, and state $x(t) \in \mathbf{R}^n$ represents the value of the different processes of the system. We model the dynamics (behavior over time) of the system using the linear equation given by

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, 1, \ldots, N-1,$$

which can be interpreted that the state of the system at time $t+1$ depends on the state $x(t)$ and input $u(t)$ at the previous time instant. Here, $A \in \mathbf{R}^{n \times n}$ and $b \in \mathbf{R}^n$ represents the system and $x(N)$ is the final state of the system after N-th time instant. The initial state of the system is assumed to be zero $x(0) = 0$.

Minimum Cost Control Problem: Given the final state $x(N) = x_{\text{final}}$ that is required to be attained by the system, we want to determine the inputs $u(0)$, $u(1), \ldots, u(N-1)$ that minimize the total cost formulated as

$$C = \sum_{t=0}^{N-1} f(u(t)),$$

where $f(u)$ is given by

$$f(u) = \begin{cases} |u|, & |u| \leq 1, \\ 2|u| - 1, & |u| \geq 1. \end{cases}$$

Formulate a linear program that determines the inputs such that the total cost is minimized.

Problem 3) (5 points)
Find necessary and sufficient conditions for $x \in \mathbf{R}^n$ to minimize a differentiable convex function f over the probability simplex, $\{x \mid \mathbf{1}^T x = 1,\ x \succeq 0\}$.

Problem 4 (10 points)
Consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{max_{i=1,2,\cdots,m}(a_i^T x+b_i)}{min_{i=1,2,\cdots,p}(c_i^T x+d_i)} \\ \text{subject to} & Fx \preceq g, \end{array}$$

with variable $x \in \mathbf{R}^n$. We assume that $c_i^T x + d_i > 0$ and $max_{i=1,2,\cdots,m}(a_i^T x + b_i) \geq 0$ for all x satisfying $Fx \preceq g$ and that the feasible set is nonempty and bounded. Show how the problem can be solved by solving *one* **LP** using Charnes-Cooper transformation (See textbook Section 4.3.2).

Problem 5) (20 points)
A company produces and sells two different products. The demand for each product is unlimited, but the company is constrained by cash availability and machine capacity.

Each unit of the first and second product requires 3 and 4 machine hours, respectively. There are 20,000 machine hours available in the current production period. The production costs are PKR 300 and PKR 200 per unit of the first and second product, respectively. The selling prices of the first and second product are pkr 600 and PKR 540 per unit, respectively. The available cash is PKR 400,000; furthermore, 45% of the sales revenues from the first product and 30% of the sales revenues from the second product will be made available to finance operations during the current period.

a) Formulate a linear programming problem that aims at maximizing net income subject to the cash availability and machine capacity limitations.

b) Solve the problem graphically to obtain an optimal solution.

c) Program the problem either using CVX or Matlab Optimization Toolbox and obtain an optimal solution.

d) Suppose that the company could increase its available machine hours by 2,000 after spending PKR 40,000 for certain repairs. Should the investment be made?

Problem 6) (15 points)
Robust Beamforming: Beamforming finds applications in signal processing and wireless communications (e.g., microphone array speech processing in acoustics, radar signal processing, medical imaging, sonar and radio astronomy). During the lectures, we designed beamformer at the receiver for single transmitter and multiple receiver configuration. In this problem, we consider a design of robust beamformer for Multiple Input Multiple Output (MIMO) systems, where we have multiple antennas both at the transmitter and receiver.

System Model: Consider a wireless communication system with transmitter equipped with T number of antennas and a receiver with R number of antennas. We have the beamforming capability both at the transmitter and receiver end. The transmitter sends a signal $x \in \mathbf{R}$ as $s = ux$, where $u \in \mathbf{R}^T$ is the transmitter beamformer. The received signal at the k-th element of the receiver is given by

$$y_k = \sum_{j=1}^{T} h_{jk} u_j \, x + n_k, \quad k = 1, 2, \ldots, L, \Longrightarrow y = Hs + n, \quad s \in \mathbf{R}^T, \; y, n \in \mathbf{R}^R.$$

where h_{jk} the channel gain between j-th transmitter antenna and k-th receiver antenna and n_k is zero-mean i.i.d. Gaussian noise with variance σ.

Received Signal: The received signal at each element is combined as a linear combination to obtain the received signal

$$r = w^T y,$$

where $w \in \mathbf{R}^R$ is the receiver beam-former.

Design Problem: We assume that the signal power is given by $E[|x|^2] = P$ and transmitted power is limited that requires $\|u\|_2^2 = 1$. Formulate an optimization problem that determines the transmitter and receiver beamformers u, w such that the signal to noise ratio at the receiver is maximized. Is this problem a convex optimization problem?

Problem 7) (20 points)
We consider a linear program given by

$$\begin{aligned} \text{minimize} \quad & c^T x && (1) \\ \text{subject to} \quad & Ax \preceq b && (2) \end{aligned}$$

Here, however, the cost vector c is random, normally distributed with mean $\mathbf{E}c = c_0$ and covariance $\mathbf{E}(c - c_0)(c - c_0)^T = \sigma$. ($A, b$ and x are deterministic). Thus, for a given $x \in \mathbf{R}^n$, the cost $c^T x$ is a (scalar) Gaussian random variable.

a) How would you minimize the expected cost $\mathbf{E}c^T x$ subject to $Ax \preceq b$?

b) In general there is a trade-off between small expected cost and small cost variance. One way to take variance into account is to minimize a linear combination

$$\mathbf{E}c^T x + \gamma \mathbf{var}(c^T x)$$

of the expected value $\mathbf{E}c^T x$ and the variance $\mathbf{var}(c^T x) = \mathbf{E}(c^T x)^2 - (\mathbf{E}c^T x)^2$. This is called the *risk-sensitive cost* and the parameter $\gamma \geq 0$ is called the *risk-aversion parameter*, since it sets the relative values of cost variance and expected value. (For $\gamma > 0$, we are willing to trade-off an increase in expected cost for a decrease in cost variance). How would you minimize the risk-sensitive cost? Is this problem a convex optimization problem? Be as specific as you can.

c) We can also minimize the risk-sensitive cost, but with $\gamma < 0$. This is called *risk-seeking*. Is this problem a convex optimization problem?

d) Another way to deal with the randomness in the cost $c^T x$ is to formulate the problem as

$$\begin{gathered} \text{minimize } \beta \\ \text{subject to } \mathbf{prob}(c^T x \geq \beta) \leq \alpha \\ Ax \preceq b. \end{gathered}$$

Here, α is a fixed parameter, which corresponds roughly to the reliability we require, and might typically have a value of 0.01. Is this problem a convex optimization problem? Be as specific as you can. Can you obtain risk-seeking by choice of α? Explain.