LUMS
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EE563 Convex Optimization

Assignment 03 Solution

Problem 1

a) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1}. \end{array}$$

in the variables x, t. To see the equivalence, assume x is fixed in this problem, and we optimize only over t. The constraints say that

$$-t \leq a_k^T x - b_k \leq t$$

for each k, *i.e.*, $t \geq |a_k^T x - b_k|$, *i.e.*,

$$t \geq \max_k |a_k^T x - b_k| = \|Ax - b\|_\infty.$$

Clearly, if x is fixed, the optimal value of the LP is $p^\star(x) = \|Ax - b\|_\infty$. Therefore optimizing over t and x simultaneously is equivalent to the original problem.

b) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -\mathbf{1} \leq x \leq \mathbf{1}, \end{array}$$

with variables $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

c) Equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y + t \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -t\mathbf{1} \preceq x \preceq t\mathbf{1}, \end{array}$$

with variables x, y, and t.

Problem 2

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Hu = x_{\text{final}} \\ & -y \preceq u \preceq y \\ & t \succeq y \\ & t \succeq 2y - \mathbf{1} \end{array}$$

where

$$H = \left[\begin{array}{ccccc} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{array}\right].$$

Problem 3

Simplex, $S = \{ y \mid \mathbf{1}^T y = 1,\ y \succeq 0 \}$ $\qquad y \in R^n, \qquad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in R^n$

First-order condition; $\nabla^T f(x)(y-x) \geq 0 \qquad \underbrace{\forall\, y \geq 0,\ \mathbf{1}^T y = 1}_{y \in S}$

Alternatively, we can write first-order condition for $y = e_i$ (standard basis vector) as

$$\nabla f(x_i) - \nabla^T f(x)\, x \geq 0 \Rightarrow \boxed{\nabla f(x_i) \geq \nabla^T f(x)\, x}$$

For an y; we have

$$\boxed{\min_{i=1,2,\dots,n} \nabla f(x_i) \geq \nabla^T f(x)\, x} \leftarrow \text{Conditions}$$

We started with first-order condition and obtained this equation. This can also be shown the other way.

Problem 4

First note the following

$$\begin{array}{ll} \text{minimize} & \frac{max_{i=1,2,\cdots,m}(a_i^T x+b_i)}{min_{i=1,2,\cdots,p}(c_i^T x+d_i)} \\ \text{subject to} & Fx \preceq g, \end{array} \quad = \quad \text{minimize} \max_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,p}} \frac{a_i^T x + b_i}{c_i^T x + d_i}$$

subject to $\quad Fx \preceq g$

Choosing $x = \frac{y}{z}$, we reformulate the problem: (z scalar)

$$\begin{array}{ll} \underset{(y,z)}{\text{minimize}} & \max_{i=1,\dots,m} a_i^T y + b_i z \\ \text{subject to} & c_j^T y + d_j z = 1, \quad j=1,\dots,p \\ & Fy - gz \preceq 0 \\ & z \geq 0 \end{array}$$

Using epigraph reformulation

$$\begin{array}{lll} \text{minimize} & s & \\ \text{subject to} & a_i^T y + b_i z \leq s, & i=1,2,\dots,m \\ & c_j^T y + d_j z = 1 & j=1,2,\dots,p \\ & Fy - gz \preceq 0 & \\ & z \geq 0 & \end{array}$$

(QED)

Problem 5

Define variables; Assume x and y represent units of first and second product respectively.

* Objective function: $300x + 340y$ (Net Income)

* Constraints:

- $3x + 4y \leq 20{,}000$ (Capacity)
- $300x + 200y - (0.45)(600x) - (0.3)(540y) \leq 400{,}000$ (Cash)

  Simplify: $0.3x + 0.38y \leq 4000$
- $x \geq 0, \quad y \geq 0$ (Implicit constraints)

LP:

minimize $\begin{bmatrix} -300 & -340 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

subject to $\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0.3 & 0.38 \\ 3 & 4 \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 4000 \\ 20000 \end{bmatrix}$

(b) $x^* = 6667, \quad y^* = 0, \quad p^* = 2{,}000{,}000$ (2 million PKR)

(d) Constraints change

$3x + 4y \leq 22000$

$0.3x + 0.38y \leq 3600$

$\Rightarrow$

$x^* = 7333$

$y^* = 0$

$p^* = 2.2$ million PKR

Since; p^* is increased, investment should be made.

Problem 6

* Transmitted signal : $s = ux$ $\quad x \in R,\ u \in R^T,\ s \in R^T$

* Received signal : $r = w^T y$

where

$$y = Hs + n = Hu + n, \quad y \in R^R$$

* $\boxed{r = w^T H u x + w^T n}$

Assume $Hu = h \in R^R$

$\Rightarrow$ $r = w^T h x + w^T n$

(Equivalent to Single Transmitter Multiple Receivers)

Optimal Beamformer (we already know) $w = \dfrac{h}{\|h\|_2}$

output SNR $= \dfrac{\|h\|^2 P}{\sigma^2}$

Optimization Problem :-

$$\text{maximize } \frac{P\|h\|^2}{\sigma^2} \equiv \text{maximize } \|h\|^2$$

$$\Rightarrow \boxed{\begin{array}{ll} \underset{u}{\text{maximize}} & \|Hu\|_2 = u^T H^T H u \\ \text{subject to} & \|u\|_2 \leq 1 \end{array}}$$

Non_convex Problem

(Limited Transmitted power)

$\Rightarrow$ We get optimal Transmitter Beamformer u^*

$$w^* = \frac{Hu^*}{\|Hu^*\|_2}$$

Problem 7

(a) minimize $c_0^T x$

subject to $Ax \preceq b,$

* LP

(b) minimize $c_0^T x + \gamma x^T \Sigma x$

subject to $Ax \preceq b$

(See Module 1; example 4)

* QP
- $\gamma \geq 0$
- Σ positive semi-definite

(c) If $\gamma < 0$, it is not a convex optimization problem

(d) This is, in fact, a Robust LP (See Module 2)

We can obtain risk-seeking by changing the value of α. Larger the value; implies greater risk.