LUMS
A Not-for-Profit University

EE563 Convex Optimization

Assignment 04 Solution

Problem 1

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m z_i^2 \\ \text{subject to} & a_i^T x - b_i = y_i, \quad i = 1, \dots, m \\ & y_i^2 \le z_i, \quad i = 1, \dots, m \end{array}$$

Problem 2

Let $y_j = x_j^2$. Because $x_j \ge 0$, we can recover x_j as $x_j = y_j^{1/2}$. The quadratic functions f_i can be written in terms of y as

$$f_i(x) = \frac{1}{2}\sum_{j=1}^n (P_i)_{jj} y_j + \frac{1}{2}\sum_{j \ne k} (P_i)_{jk} (y_j y_k)^{1/2} + \sum_{j=1}^n (q_i)_j y_j^{1/2}.$$

Since $y_j^{1/2}$ and $(y_j y_k)^{1/2}$ (the geometric mean of y_j and y_k) are concave and $(P_i)_{jk} \le 0$, $q_i \preceq 0$, this is convex in y. Thus the QCQP becomes a convex problem in y.

Problem 3

We can assume without loss of generality that $m = 2^K$ for some positive integer K. (If not, define $a_i = 0$ and $b_i = -1$ for $i = m+1, \dots, 2^K$, where 2^K is the smallest power of two greater than m.)
Let us first take $m = 4$ ($K = 2$) as an example. The problem is equivalent to

$$\begin{array}{ll} \text{maximize} & y_1 y_2 y_3 y_4 \\ \text{subject to} & y = Ax - b \\ & y \succeq 0, \end{array}$$

which we can write as

$$\begin{array}{ll} \text{maximize} & t_1 t_2 \\ \text{subject to} & y = Ax - b \\ & y_1 y_2 \ge t_1^2 \\ & y_3 y_4 \ge t_2^2 \\ & y \succeq 0, \quad t_1 \ge 0, \quad t_2 \ge 0, \end{array}$$

and also as

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & y = Ax - b \\ & y_1 y_2 \ge t_1^2 \\ & y_3 y_4 \ge t_2^2 \\ & t_1 t_2 \ge t^2 \\ & y \succeq 0, \quad t_1, t_2, t \ge 0. \end{array}$$

Expressing the three hyperbolic constraints

$$y_1 y_2 \ge t_1^2, \qquad y_3 y_4 \ge t_2^2, \qquad t_1 t_2 \ge t^2$$

as SOC constraints yields an SOCP:

$$\begin{array}{ll} \text{minimize} & -t \\ \text{subject to} & \left\| \begin{bmatrix} 2t_1 \\ y_1 - y_2 \end{bmatrix} \right\|_2 \le y_1 + y_2, \quad y_1 \ge 0, \quad y_2 \ge 0 \\ & \left\| \begin{bmatrix} 2t_2 \\ y_3 - y_4 \end{bmatrix} \right\|_2 \le y_3 + y_4, \quad y_3 \ge 0, \quad y_4 \ge 0 \\ & \left\| \begin{bmatrix} 2t \\ t_1 - t_2 \end{bmatrix} \right\|_2 \le t_1 + t_2, \quad t_1 \ge 0, \quad t_2 \ge 0 \\ & y = Ax - b. \end{array}$$

We can express the problem as

$$\begin{array}{ll} \text{maximize} & y_{00} \\ \text{subject to} & y_{K-1,j-1} = a_j^T x - b_j, \quad j = 1, \ldots, m \\ & y_{ik}^2 \le y_{i+1,2^k} y_{i+1,2^k+1}, \quad i = 0, \ldots, K-2, \quad k = 0, \ldots 2^i - 1 \\ & Ax \succeq b, \end{array}$$

where we have introduced auxiliary variables y_{ij} for $i = 0, \ldots, K-1$, $j = 0, \ldots, 2^i - 1$. Expressing the hyperbolic constraints as SOC constraints yields an SOCP.
The equivalence can be proved by recursively expanding the objective function:

$$\begin{array}{rcl} y_{00} & \le & y_{10} y_{11} \\ & \le & (y_{20} y_{21})(y_{22} y_{23}) \\ & \le & (y_{30} y_{31})(y_{32} y_{33})(y_{34} y_{35})(y_{36} y_{37}) \\ & & \cdots \\ & \le & y_{K-1,0}\, y_{K-1,1} \cdots y_{K-1,2^K-1} \\ & = & (a_1^T x - b_1) \cdots (a_m^T x - b_m). \end{array}$$

Problem 4

LP as SDP:

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b. \end{array}$$

QP as SDP:

Express $P = WW^T$ with $W \in \mathbf{R}^{n \times r}$.

$$\begin{array}{ll} \text{minimize} & t + 2q^T x + r \\ \text{subject to} & \begin{bmatrix} I & W^T x \\ x^T W & tI \end{bmatrix} \succeq 0 \\ & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b, \end{array}$$

with variables x, $t \in \mathbf{R}$.

QCQP as SDP:

Express $P_i = W_i W_i^T$ with $W_i \in \mathbf{R}^{n \times r_i}$.

$$\begin{array}{ll} \text{minimize} & t_0 + 2q_0^T x + r_0 \\ \text{subject to} & t_i + 2q_i^T x + r_i \le 0, \quad i = 1, \ldots, m \\ & \begin{bmatrix} I & W_i^T x \\ x^T W_i & t_i I \end{bmatrix} \succeq 0, \quad i = 0, 1, \ldots, m \\ & Ax = b, \end{array}$$

with variables x, $t_i \in \mathbf{R}$.

SOCP as SDP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (Ax_i + b_i)^T & (c_i^T x + d_i)I \end{bmatrix} \succeq 0, \quad i = 1, \dots, N \\ & Fx = g. \end{array}$$

Problem 5

a) Noting the fact $\lambda_1(x) \le t$ iff $F(x) \preceq tI$, we formulate the following SDP using epigraph reformulation approach:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & F(x) \preceq tI \end{array}$$

b) Using the Schur complement theorem we can write the problem as an SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & c \\ c^T & t \end{bmatrix} \succeq 0 \end{array}$$

c)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & c_i \\ c_i^T & t \end{bmatrix} \succeq 0, \quad i = 1, \dots, K. \end{array}$$

d) The cost function can be expressed as

$$f(x) = \lambda_{\max}(F(x)^{-1}),$$

so $f(x) \le t$ if and only if $F(x)^{-1} \preceq tI$. Using a Schur complement we get

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & I \\ I & tI \end{bmatrix} \succeq 0. \end{array}$$

e) The cost function can be expressed as

$$f(x) = \bar{c}^T F(x)^{-1} \bar{c} + \mathbf{tr}(F(x)^{-1} S).$$

If we factor S as $S = \sum_{k=1}^m c_k c_k^T$ the problem is equivalent to

$$\text{minimize} \quad \bar{c}^T F(x)^{-1} \bar{c} + \sum_{k=1}^m c_k^T F(x)^{-1} c_k,$$

which we can write as an SDP

$$\begin{array}{ll} \text{minimize} & t_0 + \sum_k t_k \\ \text{subject to} & \begin{bmatrix} F(x) & \bar{c} \\ \bar{c}^T & t_0 \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} F(x) & c_k \\ c_k^T & t_k \end{bmatrix} \succeq 0, \quad k = 1, \dots, m. \end{array}$$

Problem 6

a)

$$\begin{array}{ll} \text{minimize} & \max_i(\sum_{k\neq i} G_{ik}p_k + \sigma_i)/(G_{ii}p_i) \\ & 0 \leq p_i \leq P_i^{\max} \\ & \sum_{k\in K_l} p_k \leq P_l^{\text{gp}} \\ & \sum_{k=1}^{n} G_{ik}p_k \leq P_i^{\text{rc}} \end{array}$$

b) Code:

```
n = 5;
G =[1    0.1  0.2  0.1  0
    0.1  1    0.1  0.1  0
    0.2  0.1  2    0.2  0.2
    0.1  0.1  0.2  1    0.1
    0    0    0.2  0.1  1];
sigma = 0.5;
Pmax = 3;

% set up lower and upper bounds
l = 0;
u = 100;
tol = 1e-4;
Gtilde = G - diag(diag(G));

% use bisection to solve linear-fractional problem
while u-l > tol
    t = (l+u)/2;

    % solve feasibility problem for this value of t
    cvx_begin
        cvx_quiet(true);
        variable p(n);
        Gtilde*p + sigma*ones(n,1) <= t * diag(G).*p;
        p >= 0;
        p <= Pmax;
        p(1)+p(2) <= 4;
        p(3)+p(4)+p(5) <= 6;
        G*p <= 5;
    cvx_end

    if strcmpi(cvx_status, 'Solved')
        u = t;
        % save best values
        pstar = p;
        sstar = 1/t;
    else
        l = t;
    end
end

% output results
pstar
sstar
```

Optimal values:

optimal values of the trasmitted powers are: $p_1 = 2.1188$, $p_2 = 1.8812$, $p_3 = 1.6444$, $p_4 = 2.3789$, $p_5 = 1.8011$. The maximum SINR is 1.6884.