LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE563/MATH325 Convex Optimization (Spring 2020)
Quiz 02 - Solutions

Name: ______________________

Campus ID: ______________________

Total Marks: 20
Time Duration: 25 minutes

Question 1 (8 marks)

Determine whether each of the following functions defined on $\mathbf{R}^n$ or specified domain is convex or concave or neither. Provide brief justification to support your answer.

(a) $f(x) = a^T x + b, \;\; x \in \mathbf{R}^n$

Solution: Affine and therefore convex and concave.

(b) $f(x) = \frac{1}{1-x^T x}, \quad \|x\|_2 < 1, \, x \in \mathbf{R}^n$

Solution: $1 - x^T x$ is concave, $\frac{1}{x}$ is convex and non-increasing, f is a convex, non-increasing function of a concave function and is therefore convex.

(c) $f(x) = e^x + e^{-x}, \quad x \in \mathbf{R}$

Solution: Convex

(d) $f(x) = \max\{1/x, x^2, e^x\}, \quad x \in \mathbf{R}_+$

Solution: Convex; point-wise maximum of convex functions.

(e) $f(x, y, z) = -\log(yz - x^T x), \quad \mathbf{dom} f = \{x \in \mathbf{R}^n, y, z \in \mathbf{R}_{++} | \, yz > x^T x\}$

Solution: f is convex. $\log(yz - x^T x) = \log(y) + \log(z - x^T x/y)$. Since $z - x^T x/y$ is concave, $\log(z - x^T x/y)$ is concave and consequently $\log(yz - x^T x)$ is concave.

Question 2 (3 marks)

Consider a polynomial

$$P(x, \omega) = x_1 + x_2 \cos \omega + x_3 \cos 2\omega + \cdots + x_n \cos(n-1)\omega.$$

Show that the function

$$f(x) = -\int_0^{2\pi} \log P(x, \omega) d\omega,$$

is convex on $\{x \in \mathbf{R}^n | P(x, \omega) > 0, \, \omega \in [0, 2\pi]\}$.

Solution: $\log P(x, \omega)$ is concave in x since $P(x, \omega)$ is affine in x for each ω. $\int_0^{2\pi} \log P(x, \omega) d\omega$ is a non-negative weighted sum of concave functions and is therefore concave. $\int_0^{2\pi} \log P(x, \omega) d\omega$ is convex.

Question 3 (4 marks)

If $f : \mathbf{R}^n \to \mathbf{R}$ is a convex function show that its α-sub-level set S_α given by

$$S_\alpha = \{x \in \mathbf{R}^n | f(x) \leq \alpha\}$$

is convex.

Solution: For $x_1, x_2 \in S_\alpha$, we have $f(x_1) \le \alpha$ and $f(x_2) \le \alpha$. By convexity of

$$f(\theta x_1 + (1-\theta)x_2) \le \theta f(x_1) + (1-\theta) f(x_2) \le \theta\alpha + (1-\theta)\alpha \le \alpha,$$

which implies that $\big(\theta x_1 + (1-\theta)x_2\big) \in S_\alpha$.

Question 4 (5 marks)

Determine the conjugate of the function $f(x) = x^p$, $x \in \mathbf{R}_{++}, \quad p > 1$.

Solution: By definition, we have

$$f^*(y) = \sup_{x>0} (yx - x^p)$$

For $y < 0$, the supremum is zero at $x = 0$ and therefore we have $f^*(y) = 0$.
For $y > 0$, the maximum occurs at $x = \big(\frac{y}{p}\big)^{1/(p-1)}$ and

$$f^*(y) = (p-1)\Big(\frac{y}{p}\Big)^{p/(p-1)}, \quad y > 0.$$