EE563/MATH325 – Convex Optimization

Final Examination (Spring 2020)

INSTRUCTIONS:

- We require you to solve the exam in a single time-slot of four hours without any external or electronic assistance.
- We encourage you to solve the exam on A4 paper, use new sheet for each question and write sheet number on every sheet.
- Clearly outline all your steps in order to obtain any partial credit.
- The exam is closed book and notes. You are allowed to have two A4 sheet with you with hand-written notes on both sides. Calculators can be used.
- We assume $x \in \mathbf{R}^n$ unless specified otherwise.
- If you are ready, please proceed to the next page.

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Part - 1

Problem 1. (36 pts) Select ALL (upto three) correct choices. Every incorrect answer would earn a penalty of 1 point but the total marks of any multiple choice question will not be less than zero.

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(1) For an optimization problem in $\mathbf{R}^2$ given by

$$\begin{aligned} \text{minimize} \;\; & \max(x_1, x_2), \\ \text{subject to} \;\; & 2x_1 + x_2 \geq 1, \\ & x_1 \geq 0, x_2 \geq 0, \\ & x_1 + 3x_2 \geq 1, \end{aligned}$$

which of the following is/are correct.

(a) The problem has one global solution and no local solution.
(b) The problem has one global solution and one local solution.
(c) The problem has one global solution and two or more local solutions.
(d) The problem does not have any solution (Infeasible).

(2) Consider a measurement model $y = Ax + n$, that is, observations y are given by the affine transformation of the input $x \in \mathbf{R}^n$ with $A \in \mathbf{R}^{m \times n}$ plus additive noise n. Given y, x is determined by solving the following problem

$$\text{minimize} \;\; f_o(x) = \|y - Ax\|_2^2.$$

The Hessian of the objective function $f_o(x)$ with respect to x is given by

(a) $2AA^T$
(b) $2(A + A^T)$
(c) $2(A^T A)^{-1} A^T$
(d) $2A^T A$
(e) $2(A - A^T)$

(3) For an optimization problem

$$\begin{aligned} \text{minimize} \;\; & f_0(x) = x \log x \\ \text{subject to} \;\; & x > 0 \end{aligned}$$

with **dom** $f_0 = \mathbf{R}_{++}$, choose the correct statement(s)

(a) Optimization problem is convex
(b) Problem is infeasible, that is, $p^* = \infty$
(c) Problem is unbounded below, that is, $p^* = -\infty$
(d) Optimal value is achieved at unique optimal point

(4) If the feasible set of an optimization problem is unbounded then

(a) no finite optimal point exists
(b) the problem has infinite number of feasible points
(c) existence of finite optimal point is not guaranteed
(d) the problem is infeasible

(5) Consider a problem

$$\text{minimize } x + 2$$
$$\text{subject to } x^2 \leq 1,$$

where $x \in \mathbf{R}$. Choose the correct statement(s).

(a) Problem is convex
(b) Problem is feasible
(c) Optimal value is 0
(d) Optimal value is 1

(6) Consider a convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax - b = 0. \end{array}$$

The problem satisfies slater's constraint qualification and the optimal value of the problem is $p^* = 100$ for some optimal x^* such that $f_1(x^*) = -0.5$. Choose the correct option given the following statements about the problem.

- Statement 1: The dual function of an optimization problem evaluated at some (λ, μ), with $\lambda \geq 0$, is equal to 20.
- Statement 2: Dual optimal variable λ_1^* is $\lambda_1^* = -2$.

(a) Both statements are true.
(b) Statement 1 is true and statement 2 is false
(c) Statement 1 is false and statement 2 is right
(d) Both statements are false.

(7) For an optimization problem

$$\begin{array}{ll} \text{maximize} & g(x) \\ \text{subject to} & f(x) \leq 0, \end{array}$$

a set of conditions for a local maximum of $g(x)$ to be global maximum are

(a) g concave, f convex
(b) f concave, g convex
(c) g convex, f convex
(d) f concave, g concave

Information for the next two questions:
Consider a measurement model (e.g., multiple antenna receiver with k number of antenna elements) given by $y = hx + n$, that is, you have the signal observations $y \in \mathbf{R}^k$ and you wish to reconstruct $x \in \mathbf{R}$ by combining the observations linearly as $w^T y$, where $w \in \mathbf{R}^k$ represents weights (beamformer). We assume that noise $n \in \mathbf{R}^k$ is zero mean i.i.d with variance σ^2.

(8) The noise power at the output of the beamformer, that is, the power of noise in $w^T y$ is given by

(a) σ^2
(b) $\frac{\sigma^2}{w^T w}$
(c) $\sigma^2 w^T w$
(d) $\sigma^2 w w^T$
(e) $w^T w$

(9) The optimization problem to determine the optimal w that minimizes the signal-to-noise ratio (SNR) at the output of the beamformer can be formulated as

(a) LP
(b) QP
(c) SOCP
(d) SDP

(10) For an optimization (Primal) problem (not necessarily convex) in the standard form, given by

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p. \end{array}$$

If the dual problem is given by

$$\begin{array}{ll} \text{maximize} & g(\lambda, \mu) \\ \text{subject to} & \lambda \succeq 0, \end{array}$$

where $g(\lambda, \mu)$ is a Lagrange dual function, which of the following statement(s) is/are true:

(a) $g(\lambda, \mu)$ is a convex function in both λ and μ
(b) Dual problem is a convex optimization problem only when the primal problem is a convex optimization problem
(c) $g(\lambda, \mu)$ is a linear function in both λ and μ
(d) $g(\lambda, \mu)$ is a concave function in both λ and μ
(e) $g(\lambda, \mu)$ is a linear function in both λ and μ if and only if the Primal problem is a convex optimization problem

(11) Consider a convex optimization (Primal) problem in standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

with the dual problem given by

$$\begin{array}{ll} \text{maximize} & g(\lambda, \mu) \\ \text{subject to} & \lambda \succeq 0, \end{array}$$

where $g(\lambda, \mu)$ is a Lagrange dual function. If x^*, λ^* and μ^* satisfy KKT conditions, which of the following relation(s) hold for x^*, λ^* and μ^*:

(a) $\lambda^* = 0$
(b) $\lambda_i^* f_i(x^*) = 0$ for $i = 1, \ldots, m$
(c) $g(\lambda^*, \mu^*) > f_0(x^*)$
(d) $g(\lambda^*, \mu^*) < f_0(x^*)$

(12) Consider an optimization problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

where c is random with mean c and covariance Σ. An optimization problem can be formulated as

(a) LP
(b) Linear-fractional Optimization problem
(c) SOCP
(d) All of the above
(e) None of the above

Part - 2

Problem 2. (**14 pts**)
(**Independent parts**)

(a) (**5 pts**)
We consider a wireless communication system with four users being served from a common transmitter. Signal-to-noise-plus-interference ratio (SINR) for the k-th user is given by

$$\text{SINR}_k = \frac{G_k P_k}{N + \sum\limits_{i=1, i \neq k}^{4} G_i P_i}, \quad k = 1,\ 2, 3, 4.$$

Here $P_i,\ i = 1, 2, 3, 4$ and $G_i,\ i = 1, 2, 3, 4$ denote the power transmitted to i-th user and i-th user channel gain respectively. N denotes the noise power.

We consider a problem to determine the powers that minimize the total power under constraints on SINR given by $\text{SINR}_k \geq \gamma_k$ for $k = 1,\ 2, 3, 4$. Argue that the problem is convex and formulate the problem as an LP.

(b) (**4 pts**) A quasi-convex optimization problem can be solved using a bisection algorithm by iteratively solving a series of feasibility problems. If you are given the following information about the optimal value p^* of the problem:

$$p^* \leq \alpha, \quad \text{and} \quad p^* \geq \beta,$$

determine the number of iterations of the bisection algorithm required to determine ϵ-suboptimal point.

(c) (**5 pts**) Argue that the following unconstrained optimization problem is convex and formulate it as SOCP. Here $x \in \mathbf{R}^n$ is the optimization variable, $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$.

$$\text{minimize} \quad 10\|x\|_\infty + \tfrac{1}{100}\|x\|_1 + 9\|Ax + b\|_2.$$

Problem 3. (**10 pts**)

Problem Description: You wish to send an important piece of information or signal $x \in \mathbf{R}$ to your friend. There is a possibility that the information gets corrupted by the additive noise and multiplicative perturbations. To reduce the effect of noise, you encode the information and send through M different transmitters by replicating it as

$$s = ux,$$

where $u \in \mathbf{R}^M$ contains the weights assigned to x before sending it through different channels. We refer to u as **input codeword.**

Received Signal: Your friend receives N copies of the information that is corrupted or transformed along the way as follows

$$y_k = \sum_{j=1}^{T} h_{jk} u_j \, x + n_k, \quad k = 1, 2, \ldots, N, \Longrightarrow y = Hs + n, \quad s \in \mathbf{R}^M, \; y, n \in \mathbf{R}^N,$$

where h_{jk} represents the multiplicative perturbation along the route from j-th transmitter to the k-th received component and n_k is zero-mean i.i.d. Gaussian noise with variance σ.

Information Retrieval: Your friend linearly combines the received signal as an attempt to retrieve x as follows

$$r = w^T y.$$

Design Problem: Assuming the signal power is limited as $E[|x|^2] = P$ and $\|u\|_2^2 \leq 1$ and H is known, formulate an optimization problem that determines the input codeword $u \in \mathbf{R}^M$ such that the signal-to-noise ratio of r, that is, the ratio of the power of the signal component in r to the power of the noise component in r is maximized. Is this problem a convex optimization problem?

(**Hint:** First, choose $w \in \mathbf{R}^N$ such that the noise power is suppressed as a result of combining the received signal.)

Problem 4. (7 pts)

Formulate the following convex optimization problem as a semi-definite program (SDP)

$$\text{minimize} \quad \|A_0 + x_1 A_1 + \cdots + x_n A_n\|_2$$

where each $A_i \in \mathbf{R}^{p \times q}$ and $\| \cdot |_2$ represents the maximum singular value (or spectral norm).

Problem 5. (8 pts)

For a variable $x \in \mathbf{R}^n$, $F(x)$ is defined as as

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n,$$

where $F_i \in \mathbf{S}^m$. Formulate the following optimization problem as a semidefinite program (SDP)

$$\text{maximize} \quad \lambda_m(x) - \lambda_1(x),$$

where $\lambda_1(x)$ and $\lambda_m(x)$ denote the largest and smallest eigenvalues of the matrix $F(x)$ respectively.

Problem 6. (**10 pts**) This problem is related to the geometric interpretation of duality. Consider the two dimensional optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) = e^{-x_1}, \\ \text{subject to} \quad & g(x) = \frac{x_1^2}{x_2} \leq 0, \\ & x \in X, \end{aligned}$$

where $X = \{x | x_2 > 0\}$.

(a) (**2 pts**) Is the problem convex? Briefly justify your answer.

(b) (**1 pts**) What is the optimal value p^* of the problem?

(c) (**3 pts**) Plot the set $G = \{(u, t) | \exists\, x \in X, g(x) = u, f(x) = t\}$.

(d) (**2 pts**) Determine the Lagrange dual function $g(\lambda)$, where λ is a dual variable associated with $g(x)$.

(e) (**2 pts**) What is the optimal value of the dual problem and duality gap?

Problem 7. (**7 pts**) Consider an optimization (primal) problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = x^T P x \\ \text{subject to} & Ax = b, \\ & x \succeq 0, \end{array}$$

where $P \in \mathbf{S}_{++}^n$ and $A \in \mathbf{R}^{m \times n}$.

(a) Write down KKT (optimality) conditions for the problem.

(b) Show that a point x^* that satisfies the KKT conditions is an optimal solution of the primal problem, that is, $f_o(x) \geq f_o(x^*)$ for any feasible x.

Problem 8. (**8 pts**) Consider the following optimization problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Ax \preceq b \\ & \mathbf{1}^T x = 1 \end{array}$$

where $\mathbf{dom}\, f_0 = \mathbf{R}^n_{++}$.
Information: You may directly use the following information:

$$f(z) = \inf_{y>0} \big(zy - y \log y\big) = e^{z-1}, \quad z \in \mathbf{R},$$

where y is a scalar.

(a) (**3 pts**) Formulate the Lagrangian for the given optimization problem.

(b) (**4 pts**) Determine the dual function $g(\lambda, \mu)$ of the optimization problem.

(c) (**1 pts**) State the Slater's condition that guarantees zero duality gap.