EE563/MATH325 – Convex Optimization

Mid Examination 1 (Spring 2020)

March 04, 2020 05:00 pm–07:30 pm

INSTRUCTIONS:

- Do not flip this page over until told to do so.
- **The exam consists of TWO Parts.**
- **Exam need to be solved on the exam sheet. Blue answer books are only provided for rough work. Do not return them after the exam.**
- Read all the questions before you start working on the exam. Make sure that your working is neat and readable.
- Clearly outline all your steps in order to obtain any partial credit.
- *Try to recognize the easiest way to solve a problem.*
- The exam is closed book and notes. You are allowed to bring three A4 sheet with you with hand-written notes on both sides. Calculators can be used.
- Use of pencils is allowed.

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Part - 1 (24 pts)

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Problem 1. (24 pts) Select ALL (upto three) correct choices. Every incorrect answer would earn a penalty of 1 point but the total marks of any multiple choice question will not be less than zero.

(1) A set of points $X = \{x_o + \alpha v | x_o,\, v \in \mathbf{R}^n,\, \alpha > 0\}$ is

(a) affine
(b) convex
(c) cone
(d) none of the above

(2) The norm ball $B \subseteq \mathbf{R}^n$ of radius r and center $x_c \in \mathbf{R}^n$ is represented as

(a) $\{x \in \mathbf{R}^n | \|x - x_c\|_2 = r\}$
(b) $\{x \in \mathbf{R}^n | \|x - x_c\|_2 \leq r\}$
(c) $\{x \in \mathbf{R}^n | x = x_c + ru\}$
(d) $\{x \in \mathbf{R}^n | x = x_c + ru,\, \|u\|_2 \leq 1\}$

(3) In beam-forming using multi-antenna elements, the beamforming vector of weights which ensure gain of unity for the signal lie in a

(a) half-space
(b) norm cone
(c) hyperplane
(d) ellipsoid

(4) Consider a set of points in $\mathbf{R}^n$ closer to a point y than a point z. The set is

(a) half-space
(b) convex
(c) ball
(d) hyperplane

(5) Which of the following functions are log-concave on $\mathbf{R}_{++}$?

(a) e^{-5x}
(b) $x^{-2}e^{3x}$
(c) $\frac{1}{x^{-4}}$
(d) $\sqrt{x}$

(6) A function $f : \mathbf{R}^n \to \mathbf{R}$, that is twice differentiable and given that $\mathbf{dom} f$ is a convex set, is convex if

(a) $\{(x,t) | f(x) \geq t\}$ is a convex set
(b) $f(x) \leq f(y) + \nabla f(x)^T (x - y), \quad x, y \in \mathbf{dom} f$

(c) $\{x|f(x) \le \alpha\}$ is a convex set
(d) $\nabla^2 \succeq 0$

(7) The function $f(x) = x_1x_2$ with $\mathbf{dom}f = \mathbf{R}^2_{++}$ is
(a) log-convex
(b) log-concave
(c) concave
(d) quasi-convex
(e) convex
(f) quasi-concave

(8) For a random variable x and a convex function f, we have
(a) $f(E[x]) \le E[f(x)]$
(b) $f(E[x]) \le f(x)$
(c) $f(E[x]) \ge E[f(x)]$
(d) $f(z) \le E[f(x+z)]$

(9) Consider the following statements about convex functions:
(i) If $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succeq 0$ for continuous differentiable function $f : \mathbf{R}^{n+1} \to \mathbf{R}$, then x^* is a local minimum.
(ii) The point-wise minimum function of two convex functions $f_1, f_2 : \mathbf{R} \to \mathbf{R}$ is a convex function.

Choose the correct answer(s).
(a) Statement (i) is true
(b) Statement (i) is false
(c) Statement (ii) is true
(d) Statement (ii) is false

(10) Choose the correct statement(s) about log-convex and log-concave functions.
(a) The sum of two log-concave functions is log-concave
(b) The product of two log-convex functions is convex
(c) The non-negative weighted sum of two concave functions is log-concave
(d) The sum of two log-convex functions is log-convex

(11) For each of the following set $S \subset \mathbf{R}^2$, choose the set(s) which is/are convex.
(a) $S = \{x \in \mathbf{R}^2 | ||x||_1 \ge 1\}$
(b) $S = \{x \in \mathbf{R}^2 | x_1 \ge x_2\}$
(c) $S = \{x \in \mathbf{R}^2 | ||x||_2 = 10\}$
(d) $S = \{x \in \mathbf{R}^2 | |x_1| = |x_2|\}$

(12) The function

$$f(x) = x \log x, \quad \mathbf{dom}f = \mathbf{R}_{++}$$

is
(a) concave
(b) quasi-convex
(c) convex
(d) quasi-concave

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Part - 2 (76 pts)

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Problem 2. (**8 pts**) Convexity of Sets

(a) (**3 pts**) Ellipsoid in $\mathbf{R}^n$ is defined as

$$\mathfrak{E} = \{x | x_c + Au | \|u\|_2 \leq 1\}.$$

If it can also be equivalently expressed as

$$\mathfrak{E} = \{x | (x - x_c)^T P^{-1}(x - x_c) \leq 1\}, \quad P \in \mathbf{S}^n_{++},$$

determine the relation between the matrices P and A.

(b) (**5 pts**) Determine if a set of points closer to a given point than a given subset, described as:

$$C = \{x \in \mathbf{R}^n | \|x - \alpha\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}, \quad \alpha \in \mathbf{R}^n,\ S \subseteq \mathbf{R}^n,$$

is convex. Provide justification to support your answer.

Problem 3. (**8 pts**) Consider convex cones K_1 and K_2 with associated dual cones K_1^* and K_2^* respectively. Prove the following

(a) (**3 pts**) $K_1 \subset K_2$ implies $K_2^* \subset K_1^*$.

(b) (**5 pts**) Consider a proper cone $K = \{x \in \mathbf{R}^2 | x_1 \leq 0,\ x_2 \geq 0\}$.

  (i) Sketch the cone K.

  (ii) Find the minimum or minimal element(s) of the set $S = R_+^2 \cap C$ with respect to the cone K, where $C = \{x \in \mathbf{R}^2 | [2,\ 3]x \leq 6\} \cap \{x \in \mathbf{R}^2 | \|x\| \leq 2\}$ is the Euclidean norm ball in $\mathbf{R}^2$.

  (iii) Find the minimum or minimal element(s) of the convex hull of three points: $[-3, 0]^T$, $[1, 1]^T$, $[0, -1]^T$ with respect to the cone K.

Problem 4. (**10 pts**) We consider a point $x \in \mathbf{R}^n$ as

(i) 'good' if $Ax \preceq b$ for $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$.

(ii) 'optimal' if x is greater than the good point, that is, we always have a good point y such that $x \succeq y$.

(iii) 'semi-optimal' if x can be approximated with in the accuracy of 0.2 in the component-wise fashion by optimal points, that is, $\forall(i, \epsilon > 0.2)$, $\exists$ optimal y such that $|y_i - x_i| \leq \epsilon$.

Determine whether the set of semi-optimal points is convex.

Problem 5. (**12 pts**)

(a) (**4 pts**)
Determine if the function, $f : \mathbf{R}^n \to \mathbf{R}$, given by

$$f(x) = \log\left(\log \frac{1}{\sum\limits_{i=1}^{n} e^{a_i^T x + b_i}}\right), \quad \mathbf{dom}\ f = \{x \in \mathbf{R}^n | \sum_{i=1}^{n} e^{a_i^T x + b_i} < 1\},$$

is convex and/or concave. Provide brief justification to support your answer.

(b) (**6 pts**) By computing the Hessian of the geometric mean function $f : \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x) = \big(\prod_{k=1}^{n} x_k\big)^{1/n}, \quad \mathbf{dom} f = \mathbf{R}^n_{++},$$

show that f is concave.

(c) (**2 pts**) Determine if the function, $f : \mathbf{R}^n \to \mathbf{R}$, given by

$$f(x) = (a^T x + b)/(c^T x + d), \quad \mathbf{dom} f = \{x \in \mathbf{R}^n | c^T x + d > 0\}$$

is quasiconvex and/or quasiconcave. Provide brief justification to support your answer.

Problem 6. (**10 pts**) Let $\lambda_i(X)$, $i = 1, 2, \ldots, n$ denote the eigenvalues of a matrix $X \in \mathbf{S}^n$, where eigenvalues are indexed such that $\lambda_{i+1}(X) \leq \lambda_i(X)$. Determine if the following functions, with $\mathbf{dom} f = \mathbf{S}^n$, are convex and/or concave.

(a) $f(X) = \lambda_1(X)$

(b) $f(X) = \lambda_n(X)$

(c) $f(X) = \sum_{i=1}^{n} \lambda_i(X)$

(d) $f(X) = \sum_{i=1}^{k} \lambda_i(X), \quad k \leq n$

Problem 7. (**18 pts**)

(a) (**6 pts**) Consider a composition, $f = h \circ g$, given by

$$f(x) = h(g_1(x),\, g_2(x),\, \ldots,\, g_k(x)),$$

with $h : \mathbf{R}^k \to \mathbf{R}$ and $g_i : \mathbf{R} \to \mathbf{R}$ for each $i = 1, 2, \ldots, k$. Assume that $\mathbf{dom}\, h = \mathbf{R}^k$ and $\mathbf{dom}\, g_i = \mathbf{R}^n$. Derive the conditions for h and g for which f is (i) convex, (ii) concave.

(b) (**6 pts**) Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is convex and non-negative, and $g : \mathbf{R}^n \to \mathbf{R}$ is concave and positive. Show that the function $\frac{f^2}{g}$, with domain $\mathbf{dom f} \cap \mathbf{dom} g$ is convex.
Hint: Use Jensen's inequality.

(c) (**6 pts**) Considering the fact that the function of $h(X) = -\log\det X$ with $\mathbf{dom} h = \mathbf{S}^n_{++}$ is convex, show that the function

$$g(X) = n(\mathbf{tr} X)\log(\mathbf{tr} X) - (\mathbf{tr} X)\,\log\det X,$$

is convex on $\mathbf{S}^n_{++}$.
Hint: First determine the perspective function of h.

Problem 8. (**10 pts**) Beamforming finds applications in signal processing and wireless communications (e.g., microphone array speech processing in acoustics, radar signal processing, medical imaging, sonar and radio astronomy). In this problem, we consider a design of robust beamformer.

System Model: Consider a wireless communication system with two users (transmitters) and one receiver. Receiver is equipped with L number of antenna elements. We consider one transmitter as desired user (transmitting signal x) and the other user as interferer (transmitting signal s). The received signal at the k-th element of the receiver is given by

$$y_k = h_k x + g_k s + n_k, \quad k = 1, 2, \ldots, L, \Longrightarrow y = hx + gs + n, \quad y, h, g, n \in \mathbf{R}^L.$$

where h_k and g_k represents the channel gains for desired user and interferer respectively and n_k is zero-mean i.i.d. Gaussian noise with variance σ.

Received Signal: The received signal at each element is combined as a linear combination to obtain the received signal

$$r = w^T y.$$

Design Problem: In practice, the channel gain h is not known perfectly and the mismatch between the actual channel gain and the presumed channel gain is a problem. We assume that the actual channel gain vector h belongs to the set

$$C = \{h = h_c + u \,|\, \|u\| \leq \epsilon\},$$

that is, the actual h lies in ball with known center h_c and known radius ϵ.

The design problem requires us to determine the weights w such that the ratio of the signal power to the interference plus noise power is maximized while ensuring the signal x is recovered at the receiver. Formulate the design problem as an optimization problem, that is, formulate objective function and constraint functions.