Convex Optimization

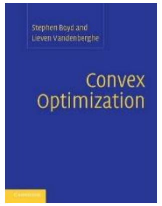
Quadratic Optimization Problems

Zubair Khalid

Department of Electrical Engineering
School of Science and Engineering
Lahore University of Management Sciences

https://www.zubairkhalid.org/ee563_2020.html

LUMS

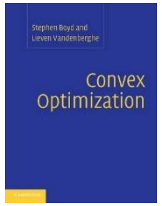


Section 4.4

Outline

- Quadratic Program (QP)
- Quadratically Constrained Quadratic Program (QCQP)
- Examples

Quadratic Program (QP)



Section 4.4

Formulation:

$f_o: R^n \to R$ $\quad\quad$ $\nabla^2 f_o(x) = P \succeq 0$

minimize $f_o(x) = \frac{1}{2}x^T P x + q^T x + r$ $\quad$ Quadratic $\quad$ (Convex) $\quad$ $P \in \mathbf{S}^n_+$

subject to $Ax \preceq b$

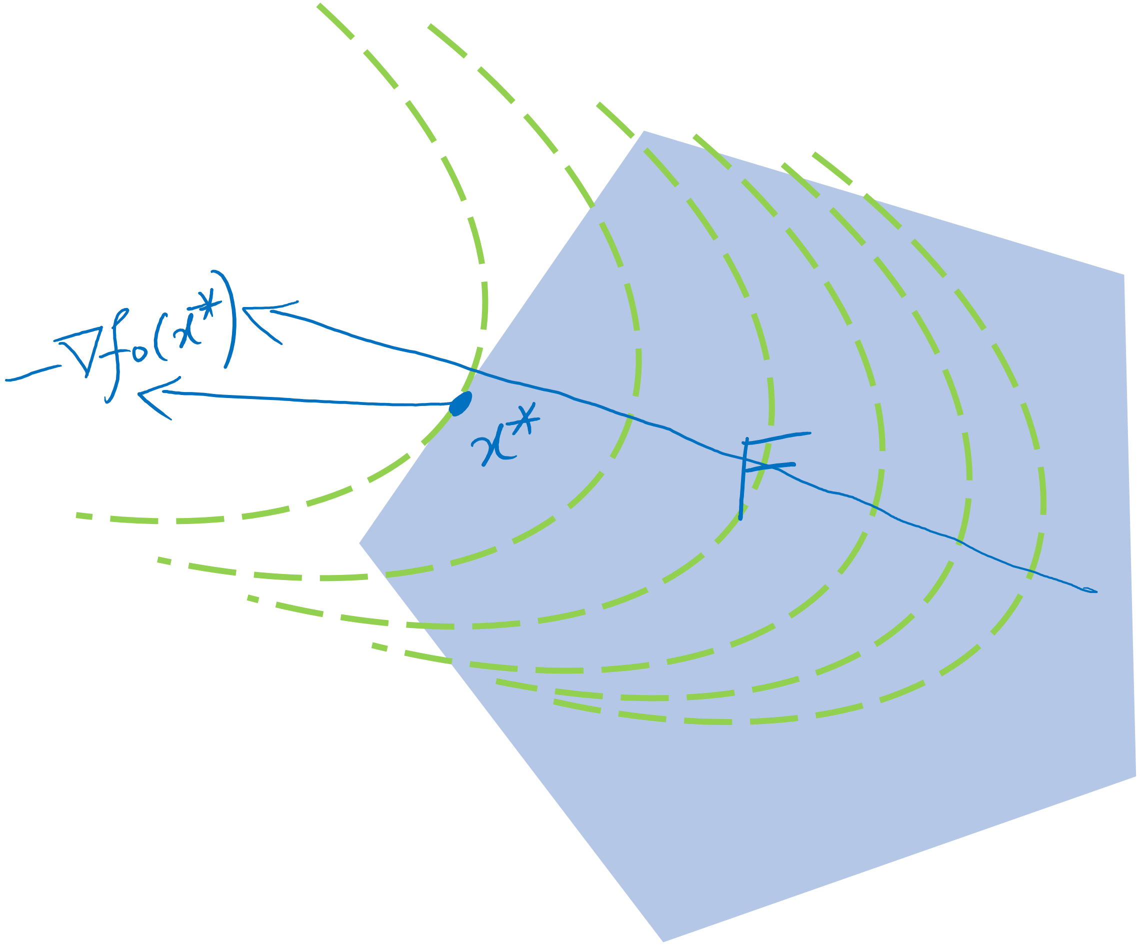
$Gx = h$

} Affine

Interpretation:

* Quadratic objective

* $F = \{x \mid Ax \preceq b, Gx = h\}$ (Polyhedron)



Quadratically Constrained Quadratic Program (QCQP)

Formulation:

$$\text{minimize} \quad f_o(x) = \tfrac{1}{2}x^T P x + q^T x + r \qquad P \in \mathbf{S}_+^n$$

$$\text{subject to} \quad \tfrac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \qquad P_i \in \mathbf{S}_+^n$$

$$Gx = h$$

(Hyper-planes)

$P_1, P_2, \ldots, P_m = 0$

QCQP → QP

$P = 0$

QCQP → LP

Interpretation:

$\boxed{\tfrac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0}$ represents an *Ellipsoid* for each *i*

$LP \subseteq QP \subseteq QCQP$

Examples

1. Linear with Quadratic Constraint

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to} \quad x^T A x \le 1, \quad A \in S^n_{++} \end{aligned}$$

$$x = A^{-1/2} y$$

$$\begin{aligned} &\text{minimize} \quad c^T A^{-1/2} y \\ &\text{subject to} \quad \underline{y^T y \le 1} \quad \left(\begin{matrix}\text{Euclidean} \\ \text{Ball}\end{matrix}\right) \end{aligned}$$

$$\text{minimize} \quad (A^{-1/2} c)^T y \leftarrow$$

$$y = \frac{-A^{-1/2} c}{\|A^{-1/2} c\|_2}$$

$$x^* = \frac{-A^{-1/2} A^{-1/2} c}{\|A^{-1/2} c\|_2} = \frac{-A^{-1} c}{\|A^{-1/2} c\|_2}$$

Examples

2. Least-Squares:

minimize $\|Ax-b\|_2^2$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$$(Ax-b)^T(Ax-b) = x^T A^T A x - 2b^T A x + b^T b$$

$x^* = A^+ b$

A^+ — Pseudo-inverse

$A^+ = (A^T A)^{-1} A^T \quad m \geq n$

$A^+ = A^T (A A^T)^{-1} \quad m \leq n$

3. Constrained Least-Squares:

minimize $\|Ax-b\|_2^2$

subject to $l \preceq x \preceq u$

$x, l, u \in \mathbb{R}^n$

Box Constraints

Examples

4. Linear Program with Random Cost

minimize $c^T x$ (Cost function)

subject to $Ax \leq b$

$Gx = h$

c — cost (Random)

$E[c] = \bar{c}$

$E[(c-\bar{c})(c-\bar{c})^T] = \Sigma$

Q; How can we incorporate randomness of c?

$\rightarrow$ Minimize $E[c^T x] = \bar{c}^T x$

$\text{var}(c^T x)$

minimize $\underline{\bar{c}^T x + \gamma \, \text{var}(c^T x)}$

$\gamma \geq 0$; sets the relative values of variance and expected value

Risk-aversion parameter

Examples

$$\mathrm{var}(c^T x) = E\left[\left(c^T x - E[c^T x]\right)^2\right] = E\left[(c^T x - \bar{c}^T x)(c^T x - \bar{c}^T x)\right]$$

$$= x^T E\left[(c - \bar{c})(c - \bar{c})^T\right] x = x^T \Sigma x$$

$$\begin{aligned} \text{minimize} \quad & \bar{c}^T x + \gamma x^T \Sigma x \quad \text{(Convex Quadratic)} \\ \text{subject to} \quad & \left.\begin{aligned} Ax &\preceq b \\ Gx &= h \end{aligned}\right\} \text{Affine} \end{aligned}$$

QP

LUMS
A Not-for-Profit University

Feedback: Questions or Comments?

Email: zubair.khalid@lums.edu.pk

Slides available at: https://www.zubairkhalid.org/ee563_2020.html

(Let me know should you need latex source)