Convex Optimization

Signal Processing Application: Compressive Sensing

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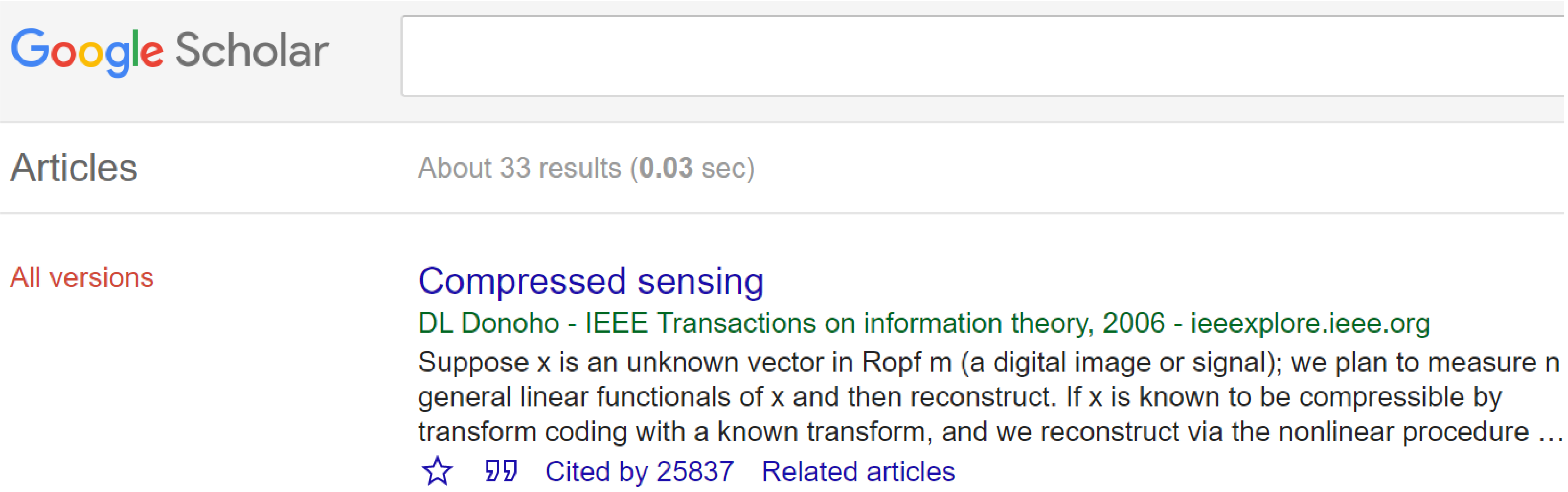
Outline

- Compressive Sensing (CS) Overview
- CS Problem Formulation
- Orthogonal Matching Pursuit (OMP) Algorithm
- ℓ_1 norm minimization

Compressive Sensing - Overview

Sparse Sampling

- Used in a variety of applications in science, engineering and beyond; image processing, computer vision, cosmology, geophysics, finance, economics, acoustics and wireless communication, to name a few.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Articles

About 33 results (**0.03** sec)

All versions

Compressed sensing

DL Donoho - IEEE Transactions on information theory, 2006 - ieeexplore.ieee.org

Suppose x is an unknown vector in Ropf m (a digital image or signal); we plan to measure n general linear functionals of x and then reconstruct. If x is known to be compressible by transform coding with a known transform, and we reconstruct via the nonlinear procedure …

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Compressive Sensing - Overview

* $1920 \times 1080 \times 3$

  $\approx \underline{6.22 \text{ MB}}$

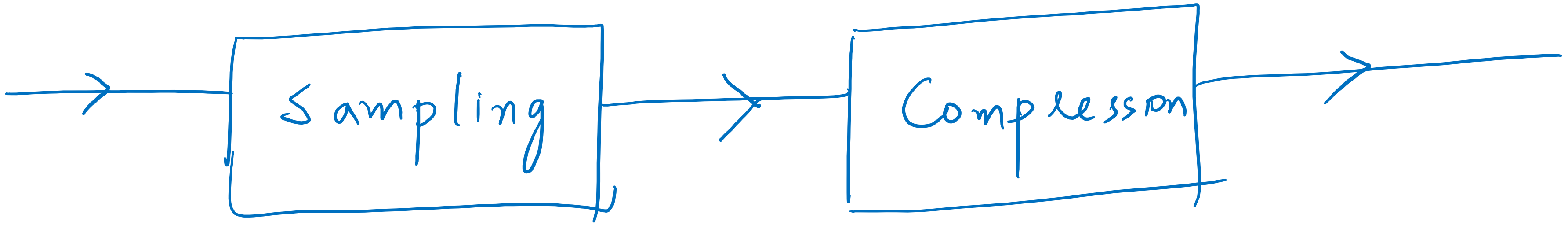
* $300 \sim 400$ KB

Q; How can we store?

A:) compression

1920×1080 8611 m

Compressive Sensing - Overview



Classical Paradigm

* Compression while sensing.

Compressive Sensing - Formulation

Problem under consideration:

* $x \in \mathbb{R}^n$
* $y = \underbrace{\Phi}_{\text{Sensing Matrix}} x$ $\qquad \Phi \in \mathbb{R}^{m \times n}$

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_m \end{bmatrix} \qquad \boxed{y_i = \phi_i^T x}$$

$m \geq n$

* Determine x given y?

Compressive Sensing - Formulation

Assumptions:

* ϕ is random

* x is sparse

$x \in R^{12}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

K-sparse

* $\boxed{y = \phi x}$

ϕ: random

x: sparse

$x = \psi u$

$y = \phi \psi u = \tilde{\phi} u$

$\boxed{y = \tilde{\phi} u}$

u: sparse

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Compressive Sensing - Formulation

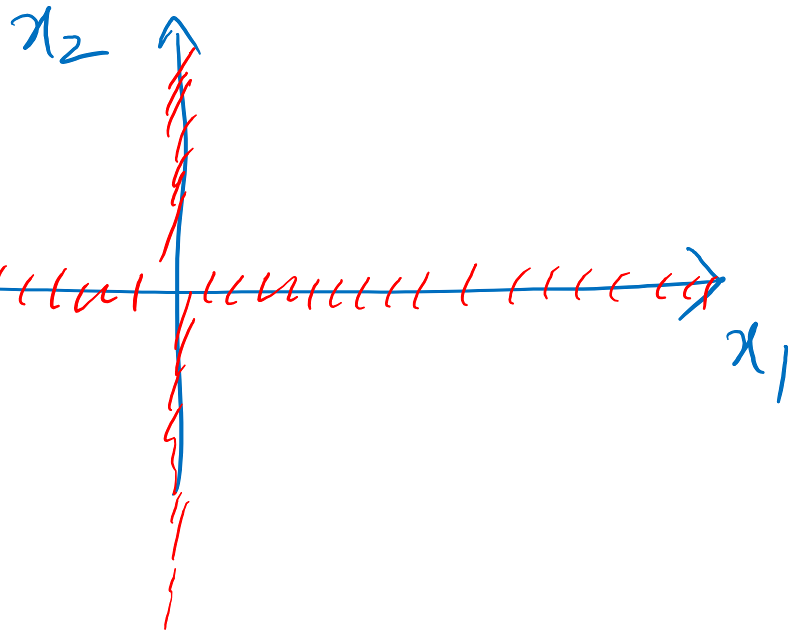
Problem Formulation:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \|x\|_0 \qquad \text{Non convex}$$

$$\text{subject to} \quad \underline{y = \Phi x}$$

$$\|x\|_0 = 1$$

$$\sum_{k=0}^{n} \binom{n}{k} \simeq O(2^n)$$



Orthogonal Matching Pursuit

Google Scholar

Articles

About 35 results (**0.05** sec)

All versions

Signal recovery from random measurements via orthogonal matching pursuit
JA Tropp, AC Gilbert - IEEE Transactions on information theory, 2007 - ieeexplore.ieee.org
This paper demonstrates theoretically and empirically that a greedy algorithm called orthogonal matching pursuit (OMP) can reliably recover a signal with m nonzero entries in dimension d given O (m ln d) random linear measurements of that signal. This is a massive …
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* stage-wise

* CS OMP

* Robust variants OMP

Orthogonal Matching Pursuit (OMP)

$$y = \phi x$$

$$\phi = \begin{bmatrix} | & | & \\ \phi_1 & \phi_2 & \cdots \\ | & | & \end{bmatrix}$$

$$y^{(1)} = y, \quad k = 1$$

<u>Step 1</u>

$$\lambda(k) = \max_j \phi_j^T y^{(k)}$$

<u>Step 2</u>

$$A = \begin{bmatrix} \phi_\lambda \end{bmatrix}$$

<u>Step 3</u>

Solve $\| y - A x^{(k)} \|_2$

$$x^{(k)} = (A^T A)^{-1} A^T y$$

<u>Step 4</u>

$$r^{(k)} = y - A x^{(k)}$$

$$y^{(k+1)} = r^{(k)}$$

Orthogonal Matching Pursuit (OMP)

$$\| r_K - r_{K-1} \|_2 \leq \epsilon_0$$

Break

$$\| y - A x^{(K)} \|_2 \leq \epsilon_0$$

After *K* iterations;

* $\lambda \in R^K$
* $A \in R^{m \times K}$
* $x^{(K)} \in R^K$

$$x = \begin{bmatrix} 0 \\ 0 \\ x_1^{(K)} \\ 0 \\ 0 \\ x_2^{(K)} \\ \vdots \end{bmatrix}$$

$\lambda(1)$

$\lambda(2)$

ℓ_1 Norm Minimization

$$\text{minimize} \quad \|x\|_0$$
$$\text{subject to} \quad y = \phi x$$

$$\text{minimize} \quad \|x\|_1 \quad \longrightarrow \text{convex}$$
$$\text{subject to} \quad y = \phi x$$

ℓ_1 Norm Minimization

$$\|x\|_1 \triangleq |x_1| + |x_2| + \cdots + |x_n|$$

$$= t_1 + t_2 + \cdots t_n = 1^T t$$

$$\begin{aligned} \text{minimize} \quad & 1^T t && \text{Linear} \\ \text{subject to} \quad & y = \Phi x && \text{(Affine)} \\ & |x_i| = t_i && i = 1, 2, \ldots n \end{aligned}$$

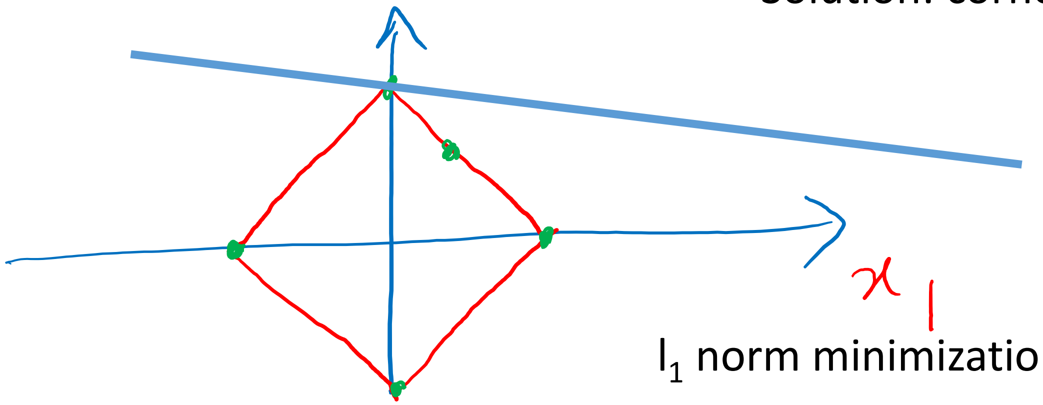
Affine Inequality

$$\begin{cases} x_i \leq t_i & i = 1, 2, \ldots n \\ -x_i \leq t_i & i = 1, 2, \ldots, n \end{cases}$$

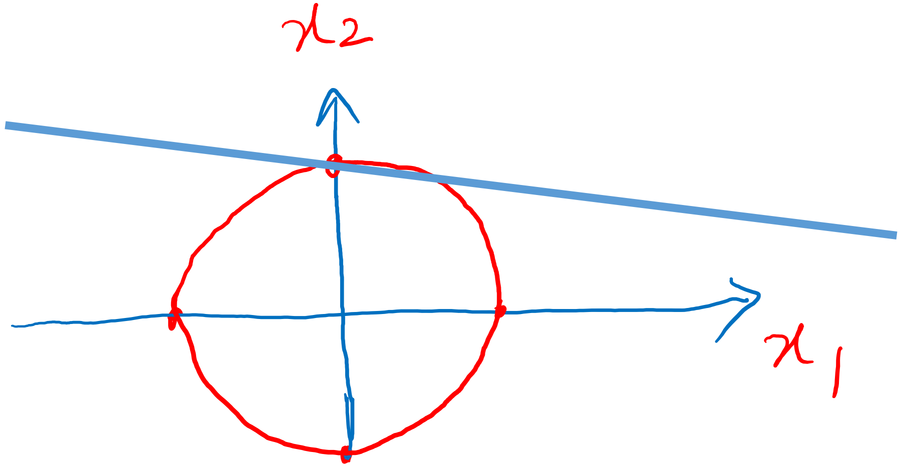
LP

ℓ_1 Norm Minimization

Solution: corner point



l_1 norm minimization enforces support



Solution: does not need to be corner point

l_2 norm minimization does not give us l_0 norm solution

ℓ_1 Norm Minimization

$$y = \phi x + n$$

$$\text{minimize} \quad \|y - \phi x\|_2 + \lambda \|x\|_1$$

Basis Pursuit Denoising

$\lambda \rightarrow$ Regularization parameter

Feedback: Questions or Comments?

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Slides available at: https://www.zubairkhalid.org/ee563_2020.html
(Let me know should you need latex source)