Convex Optimization

Second-Order Cone Programming

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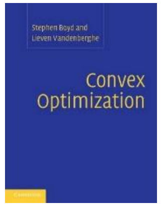
Outline



Section 4.4.2

- SOCP Formulation
- QCQP as SOCP
- Example: Robust Linear Program

Second-Order Cone Programming (SOCP)

Formulation:

$$\text{minimize} \quad f^T x + e$$

$$\text{subject to} \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i \qquad i = 1, 2, \ldots, m$$

$$Gx = h$$

Here:

$$x \in \mathbf{R}^n$$

$$A_i \in \mathbf{R}^{k_i \times n}, \quad b \in \mathbf{R}^{k_i}$$

$$G \in \mathbf{R}^{p \times n}, \quad h \in \mathbf{R}^p$$

Second-order Cone Constraint:

Let's look at the inequality constraint:

$$\|Ax + b\|_2 \leq c^T x + d \qquad A \in \mathbf{R}^{k \times n}$$

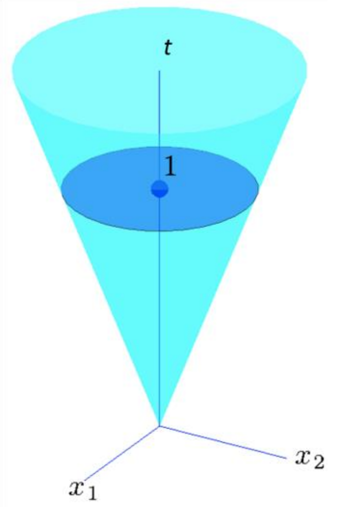
This can be interpreted as if the vector

$$\begin{bmatrix} Ax + b \\ c^T x + d \end{bmatrix} \in \mathbf{R}^{k+1}$$

lies in a second-order (Quadratic, Lorentz) cone.

In some texts, the constraint may also be expressed as:

$$(Ax + b, c^T x + d) \succeq_Q 0$$



Recall:

$$k = 2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ t \end{bmatrix} \iff \|x\|_2 \leq t$$

Second-Order Cone Programming (SOCP)

QCQP – a special case of SOCP

SOCP:

$$\begin{array}{ll} \text{minimize} & q^T x + r \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{array}$$

Let's look at the inequality constraint:

$$\|Ax + b\|_2 \leq c^T x + d \qquad A \in \mathbf{R}^{k \times n}$$

When c=0, the constraint becomes

$$\|Ax + b\|_2 \leq d \qquad \text{Obviously } d \geq 0$$

This is a quadratic constraint; squaring it converts it into Quadratic constraint. SOCP becomes QCQP.

QCQP:

variable x

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} x^T P x + q^T x + r \\ \text{subject to} & \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{array}$$

Using epigraph reformulation:

variable x, c

$$\begin{array}{ll} \text{minimize} & c \\ \text{subject to} & \frac{1}{2} x^T P x + q^T x + r \leq c \\ & \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{array}$$

Second-Order Cone Programming (SOCP)

QCQP – a special case of SOCP

In QCQP reformulation, we have constraints:

$$\frac{1}{2}x^TPx + q^Tx + r \leq c$$

$$\frac{1}{2}x^TPx + q^Tx + r \leq 0$$

Q: Are they equivalent?

$$\frac{1}{2}x^TPx + q^Tx + r - c \leq 0$$

By defining

$$\tilde{x} = \begin{bmatrix} x \\ c \end{bmatrix} \qquad \tilde{q} = \begin{bmatrix} q \\ -1 \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}\tilde{x}^T\tilde{P}\tilde{x} + \tilde{q}^T\tilde{x} + r \leq 0$$

So, they are equivalent

Q: How can we represent Quadratic constraint as second-order constraint?

$$\frac{1}{2}x^TPx + q^Tx + r \leq 0$$

$$x^TPx + 2q^Tx + 2r \leq 0$$

$$x^TB^TBx + 2q^Tx + 2r \leq 0$$

Or equivalently can berepresented as second-order constraint

$$\left\| \begin{bmatrix} Bx \\ \frac{1}{2} + q^Tx + r \end{bmatrix} \right\|_2^2 \leq \left(\frac{1}{2} - q^Tx - r\right)^2$$

$$\|Bx\|_2^2 + \left(\frac{1}{2} + q^Tx + r\right)^2 - \left(\frac{1}{2} - q^Tx - r\right)^2 \leq 0$$

$$x^TB^TBx + 4\left(\frac{1}{2}\right)\left(q^Tx + r\right) \leq 0$$

$$LP \subseteq QP \subseteq QCQP \subseteq SOCP$$

Example: Robust Linear Program

Linear Program:

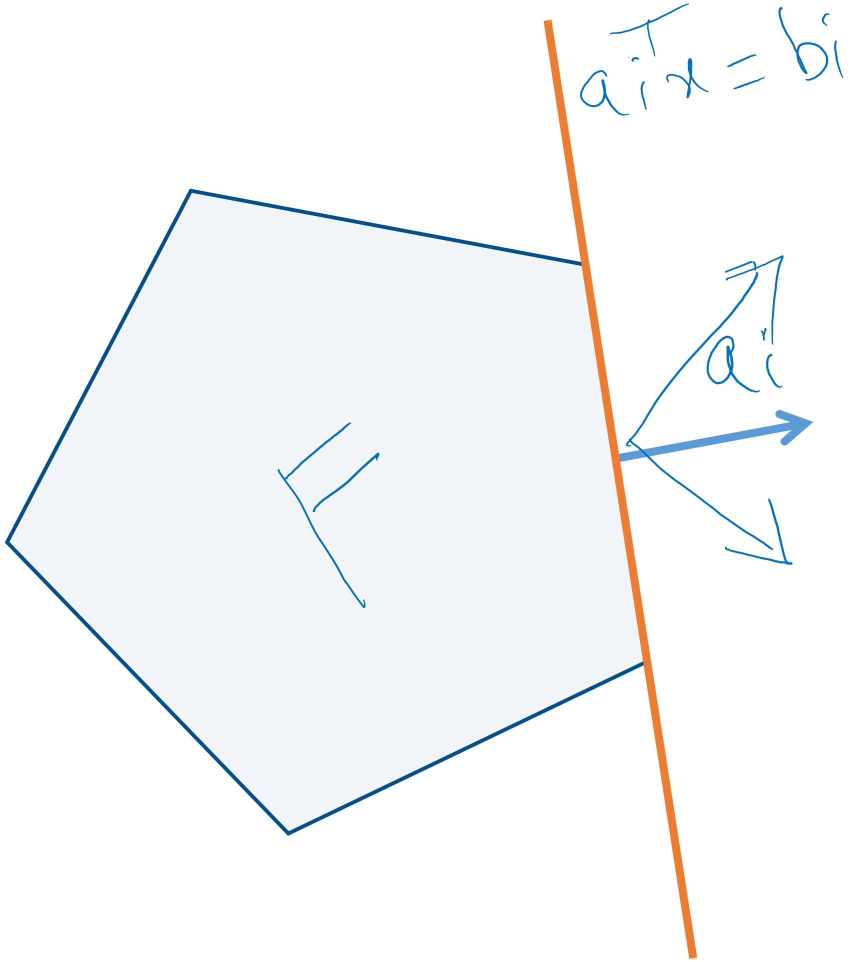
$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & a_i^T x \le b_i, \quad i = 1, 2, \ldots, m \\
& Gx = h
\end{aligned}
$$

Problem Under Consideration: Uncertainty in the constraints

We have uncertainty in a_i.

- **Deterministic Model:**
  $a_i \in \varepsilon_i = \{\bar{a}_i + A_i u \,|\, \|u\|_2 \le 1\}$ (Already studied)

- **Probabilistic Model:**
  a_i is independent Gaussian with mean $\bar{a}_i$ and covariance Σ_i.



Example: Robust Linear Program

- **Probabilistic Model:**
  a_i is independent Gaussian with mean $\bar{a}_i$ and covariance Σ_i.

Required: each constraint must hold with probability at-least η, that is

$$P(a_i^T x \leq b_i) \geq \eta, \quad i = 1, 2, \ldots, m.$$

Q: How can we incorporate this probabilistic constraint in the formulation of optimization problem?

A: This can be formulated as a second-order constraint.

Example: Robust Linear Program

$P(a_i^T x \le b_i) \ge \eta, \quad i = 1, 2, \ldots, m.$

η ; Probability ≥ 0.5

Assume

* $y = a_i^T x$
* $E[y] = E[a_i^T x] = \bar{a}_i^T x$
* $var(y) = var(a_i^T x) = x^T \Sigma_i x$

$\sigma^2 = var(y)$

$\sigma = \| \Sigma_i^{1/2} x \|_2$

Also define

$Z = \dfrac{y - E[y]}{\sigma}$; $Z \sim N(0, 1)$

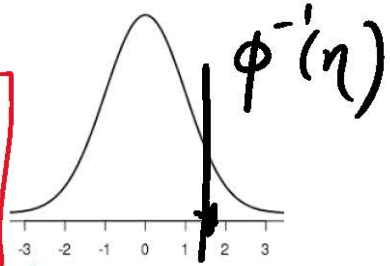
$P(a_i^T x \le b_i) = P(y \le b_i)$

$= P\left(\dfrac{y - E[y]}{\sigma} \le \dfrac{b_i - E[y]}{\sigma} \right)$

$= P\left(Z \le \dfrac{b_i - E[y]}{\sigma} \right) \geq \eta$

$\Phi\left(\dfrac{b_i - E[y]}{\sigma} \right) \geq \eta$

$\Rightarrow \boxed{\dfrac{b_i - E[y]}{\sigma} \geq \Phi^{-1}(\eta)}$



Example: Robust Linear Program

$$P(a_i^T x \le b_i) \geq \eta$$

$$\frac{b_i - a_i^T x}{\|\Sigma^{1/2} x\|_2} \geq \phi^{-1}(\eta)$$

$$\boxed{\phi^{-1}(\eta)\, \|\Sigma^{1/2} x\|_2 \le b_i - a_i^T x}$$ SOC

Probabilistic constraint as second-order constraint

SOCP

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && \phi^{-1}(\eta)\, \|\Sigma^{1/2} x\|_2 \le b_i - a_i^T x \quad i = 1, 2, \ldots, m \\ & && Gx = h \end{aligned}$$

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Feedback: Questions or Comments?

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Slides available at: https://www.zubairkhalid.org/ee563_2020.html
(Let me know should you need latex source)