Convex Optimization

QP, QCQP and SOCP - Examples

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Examples

1. Robust Quadratic Program

QP:

minimize $f_o(x) = \frac{1}{2}x^T P x + q^T x + r$

subject to $Ax \preceq b$ $Gx = h$

Problem: * Uncertainties in P
* P is subject to errors
* $P \in Y$

Robust QP :-

minimize $f_o(x) = \sup_{P \in Y} \frac{1}{2}x^T P x + q^T x + r$

subject to $Ax \preceq b$ $Gx = h$

Q; For $Y = \{P_1, P_2, \dots, P_K\}$, formulate tractable optimization problem.

* Since Y is a set of finite no. of matrices;

minimize $f_o(x) = \max_{i=1,2,\dots,K} \frac{1}{2}x^T P_i x + q^T x + r$

subject to $Ax \preceq b$ $Gx = h$

* Using epigraph reformulation t

$$\underbrace{\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \frac{1}{2}x^T P_i x + q^T x + r \le t, \; i = 1, \dots, K \\ & Ax \preceq b, \; Gx = h \end{array}}_{QCQP}$$

Examples

2. Sum of Norms

$$\underset{x}{\text{minimize}} \sum_{i=1}^{p} \|G_i x + h_i\|_2, \quad G_i \in R^{k_i \times n}$$

* Using epigraph reformulation

$$\underset{(x,t)}{\text{minimize}} \quad \sum_{i=1}^{p} t_i$$

$$\text{subject to} \quad \|G_i x + h_i\|_2 \leq t_i$$

* Constraint: Second-order

* Sum of Norms problem $\longrightarrow$ SOCP

SOC Form:

$$\|Ax + b\|_2 \leq c^T x + d$$

How can we express $\|G_i x + h_i\|_2 \leq t_i$ in this form?

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix} \in R^{n+p}$$

$$A = \left[\, G_i \;\middle|\; 0, \dots 0 \,\right], \quad b = h_i$$

$$c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad c^T \tilde{x} = t_i \qquad d = 0$$

$(n+i)$th entry is 1

Examples

3. Maximum of Norms

$$\underset{x}{\text{minimize}} \quad \max_{i=1,\ldots,K} \|G_i x + h_i\|_2, \qquad G_i \in \mathbb{R}^{k_i \times n}$$

* Using epigraph reformulation

$$\underset{x,t}{\text{minimize}} \quad t$$

$$\text{subject to} \quad \max_{i=1,\ldots,K} \|G_i x + h_i\|_2 \leq t, \qquad i = 1, \ldots, K$$

Examples

4(a). Formulation of Hyperbolic constraints as Second-order constraints

4(b). Maximizing Harmonic Mean

Hyperbolic Constraint :-

$w^2 \leq yz \qquad w \in R$

$y, z \in R_+$

$$\left\| \begin{matrix} 2w \\ y-z \end{matrix} \right\|_2 \leq y+z$$

Equivalence:

$$(2w)^2 + (y-z)^2 \leq (y+z)^2$$

$$4w^2 \leq (y+z)^2 - (y-z)^2$$

$$\Rightarrow 4w^2 \leq 4yz \Rightarrow \boxed{w^2 \leq yz}$$

Vector Case

$w^T w \leq yz \qquad w \in R^n,$

$y, z \in R_+$

equivalent to:

$$\left\| \begin{matrix} 2w \\ y-z \end{matrix} \right\|_2 \leq y+z$$

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Examples

4(b). Maximizing Harmonic Mean

$$\frac{n}{\underbrace{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}_{\text{Harmonic mean}}}$$

* maximize $\left(\sum_{i=1}^{K} \frac{1}{a_i^T x - b_i} \right)^{-1}$

subject to $Ax \geq b$

* minimize $\sum_{i=1}^{K} \frac{1}{a_i^T x - b_i}$

subject to $Ax \geq b$

* minimize $\sum_{i=1}^{K} t_i$

subject to $\frac{1}{a_i^T x - b_i} \leq t_i, \; i = 1, \ldots K$

$Ax \geq b$

* $\frac{1}{a_i^T x - b_i} \leq t_i$

* $1 \leq (t_i)(a_i^T x - b_i)$

$$\underbrace{\left\| \begin{matrix} 2 \\ a_i^T x - b_i - t_i \end{matrix} \right\|_2 \leq a_i^T x - b_i + t_i}_{\text{SOC}}$$

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Feedback: Questions or Comments?

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Slides available at: https://www.zubairkhalid.org/ee563_2020.html

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