Convex Optimization

Conic Optimization and Semidefinite Programming

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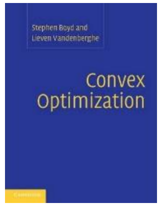
LUMS

Outline

- Convex Optimization with Generalized Inequality Constraints
- Conic Optimization Problems
- Semidefinite Programming (SDP)



Section 4.6

Convexity w.r.t. Generalized Inequality

Generalized Inequality:

For a proper cone $K \subseteq R^m$, we define a generalized inequality:

$x, y \in R^m \qquad x \preceq_K y \quad$ iff $\quad y - x \in K$

$z \in R^m \qquad 0 \preceq_K z \quad$ iff $\quad z \in K$

Convexity w.r.t. Generalized Inequality:

A function $f: R^n \to R^m$ is K_convex, where $K \subseteq R^m$ is a proper cone, if

$$f(\theta x + (1-\theta) y) \preceq_K \theta f(x) + (1-\theta) f(y)$$

for $\theta \in [0,1]$ and $x, y \in \operatorname{dom} f$,

Convex Optimization with Generalized Inequality Constraints:

Ordinary convex optimization problem

$$\begin{aligned} \text{minimize} \quad & f_o(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{aligned}$$

- *If we allow the constraint functions to be vector valued:*

$$\boxed{\begin{aligned} \text{minimize} \quad & f_o(x) \\ \text{subject to} \quad & f_i(x) \preceq_{K_i} 0, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{aligned}}$$

Convex Optimization with Generalized Inequality Constraints

$f_0 : R^n \to R$

$f_i : R^n \to R^{p_i}$

$\downarrow$

K_i convex, $K_i \subseteq R^{p_i}$

(Proper Cone)

* If $p_i = 1$, $K_i = R_+$ for each $i = 1, \ldots, m$ we have an ordinary optimization problem.

Convex Optimization with Generalized Inequality Constraints:

Properties and Interpretation

* Feasible set — convex

$$F = \{ x \mid f_i(x) \preceq_{K_i} 0,\ i = 1, 2, \dots, m,\ Gx = h \}$$

* optimal set — convex
* Locally optimal point is globally optimal point.
* Optimality Criterion: $\nabla f_0^T(x)(y - x) \geq 0 \ \forall\ y \in F$

$\Leftrightarrow x \in F$ is optimal

Conic Form Problems

In convex Optimization with Generalized Inequality Constraints, *if we consider*

- Objective function: Affine (or linear)
- Inequality constraints: Affine

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_i x - b_i \preceq_{K_i} 0, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{array}$$

Conic Form Problem or Conic Optimization Problem or Conic Program (CP)

- *Conic form problems; generalization of LP*

- The cones which we have a tractable conic optimization are of three basic types.

- K - Non-negative orthant
- K - Quadratic, Second-order
- K - Positive semidefinite

Conic Form Problems

- K_i - Non-negative orthant

$$A_i x - b_i \preceq_{K_i} 0 \quad \rightarrow \quad A_i x - b_i \preceq 0$$

  - Component-wise inequality
  - Linear inequality constraints - half-spaces
  - Conic optimization problem becomes a linear program

- K - Quadratic, Second-order

  - We have SOCP

$$\begin{array}{ll} \text{minimize} & f^T x + e \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i \quad i = 1, 2, \ldots, m \\ & Gx = h \end{array}$$

<u>Formulation of second-order constraint as conic constraint:</u>

$$\|Ax + b\|_2 \leq c^T x + d \qquad A \in \mathbf{R}^{k \times n}$$

equivalently;

$$(Ax + b, c^T x + d) \succeq_K 0$$

$$-(Ax + b, c^T x + d) \preceq_K 0$$

Semidefinite Program (SDP)

- K - Positive semidefinite
  - Conic optimization problem is referred to as *Semidefinite Program (SDP)*

* Linear Matrix Inequality (LMI)

For $x \in \mathbb{R}^n$, $A_1, A_2, \dots, A_n, B \in S^p$,

$$\underbrace{x_1 A_1 + x_2 A_2 + \cdots + x_n A_n}_{A(x)} \preceq B$$

$$A(x) \preceq B \Rightarrow B - A(x) \succeq 0$$

$$A(x) - B \preceq 0$$

* If $A_1, \dots, A_n, B$ scalars $a_1, \dots a_n, b$

$$a^T x - b \leq 0$$

* $S = \{x \mid A(x) - B \preceq 0\}$ is convex, since it is an inverse image of PSD cone under affine function: $B - A(x)$

Semidefinite Program (SDP)

Formulation:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && A_i(x) - B_i \preceq 0, \quad i = 1, 2, \ldots, m \\ &&& Gx = h \end{aligned}$$

Notes:

- If matrices defining any LMI, the LMI reduces to a linear inequality constraint (half-space).
- Multiple LMI constraints can be formulated as a single LMI.

$$A_i(x) - B_i \preceq 0, \quad i = 1, 2, \ldots, m \quad \rightarrow \quad \mathrm{diag}\Big(A_1(x) - B_1, A_2(x) - B_2, \ldots, A_m(x) - B_m\Big) \preceq 0$$

Block Diagonal Matrix

Semidefinite Program (SDP)

SOCP as SDP:

* Schur Complement

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \quad \text{(symmetric)}$$

$$S = C - B^T A^{-1} B$$

* $X \succ 0$ iff $A \succ 0$ and $S \succ 0$

* If $A \succ 0$, $X \succeq 0 \iff S \succeq 0$

Semidefinite Program (SDP)

SOCP as SDP:

SOC as LMI:

$$\|Ax + b\|_2 \leq c^T x + d \qquad A \in \mathbf{R}^{k\times n}$$

$$\begin{bmatrix} c^T x + d & (Ax+b)^T \\ Ax + b & c^T x + d \end{bmatrix} \succeq 0$$

$$\|x\|_2 \leq t$$

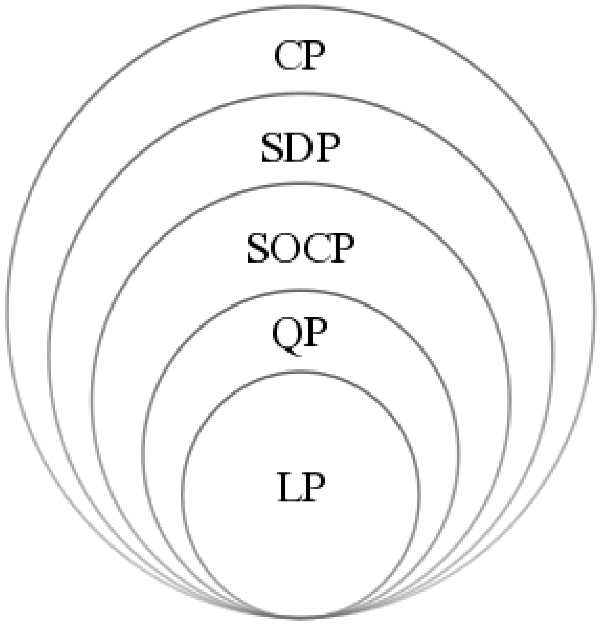
$$\left[\begin{array}{c|ccc} t & x_1 & \cdots & x_n \\ \hline x_1 & t & & \\ \vdots & & \ddots & \\ x_n & & & t \end{array}\right] = \begin{bmatrix} t & x^T \\ x & tI_n \end{bmatrix} \succeq 0$$

$$S = tI_n - \frac{x^T x}{t} \succeq 0$$

$$\Rightarrow \quad x^T x \leq t^2$$

$$\|x\|_2 \leq t$$

Big Picture: Connecting the Dots



Convex Optimization with Generalized Inequality Constraints

$$\begin{array}{ll} \text{minimize} & f_o(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{array}$$

Conic Form Problem or Conic Program (CP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_i x - b_i \preceq_{K_i} 0, \quad i = 1, 2, \ldots, m \\ & Gx = h \end{array}$$

Generalization of LP

- K - Non-negative orthant
  - LP

- K - Second-order
  - SOCP
  - QCQP
  - QP
  - LP

- K - Positive semidefinite
  - SDP
  - SOCP
  - QCQP
  - QP
  - LP

Feedback: Questions or Comments?

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Slides available at: https://www.zubairkhalid.org/ee563_2020.html

(Let me know should you need latex source)