Convex Optimization

Duality: Introduction and Lagrange Dual Function

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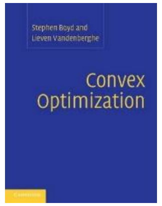
Outline

- Duality
- Lagrangian
- Lagrange Dual Function



Section 5.1

Concept of Duality

- Formalizes the concept of Lagrange multipliers

- Optimization Problems has two perspectives:
  - Original (Primal) Problem
  - Dual Problem

- The dual problem is always convex even when the primal problem is not

- The solution to the dual problem serves as a lower bound to the solution of the primal problem

- Duality Gap: the difference between the two solutions

- Duality gap is zero when the primal problem is convex and satisfy a constraint qualification

Lagrangian

Optimization problem

(Primal or Original Problem)

$$\begin{aligned} \text{minimize} \quad & f_o(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & h_j(x) = 0, \quad , j = 1, 2, \ldots, p \end{aligned}$$

(Does not need to be convex)

* optimal value = p^*

Lagrangian

$$L(x, \lambda, \mu) = f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

* These weights

λ_i ; i-th inequality constraint

μ_j ; j-th equality constraint

$\lambda \in R^m$

$\mu \in R^p$

Lagrange Multipliers
Dual variables
Penalty variables
Price variables

Lagrangian

$$L(x, \lambda, \mu) = f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

Interpretation and Notes:

- Lagrangian function appends the objective function and inorporates the constraint function as a weighted sum
- Domain of Lagrangian function: $D \times \mathbf{R}^m \times \mathbf{R}^p$
- Lagrangian is an affine function of λ and μ

$$f(x) = [f_1(x), f_2(x), \ldots, f_m(x)]$$

$$h(x) = [h_1(x), h_2(x), \ldots, h_p(x)]$$

$$L(x, \lambda, \mu) = f_o(x) + \lambda^T f(x) + \mu^T h(x)$$

Lagrange Dual Function

$$g(\lambda, \mu) = \inf_{x \in D} L(x, \lambda, \mu)$$

$$g(\lambda, \mu) = \inf_{x \in D} f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

D – domain of primal problem

Interpretation and Notes:

- $g(\lambda, \mu) = -\infty$ if $L(x, \lambda, \mu)$ is unbounded below
- $g(\lambda, \mu)$ is a concave function of (λ, μ)

How?

Since it is a pointwise infimum of affine functions

Lagrange Dual Function

* Lagrange dual function serves as a lower bound on the optimal value of the primal problem for $\lambda \succeq 0$

$$g(\lambda, \mu) \leq p^*, \qquad \lambda \succeq 0$$

p^* _ optimal value of the primal problem

How ?

Lagrange Dual Function

$$g(\lambda, \mu) = \inf_{x \in D} L(x, \lambda, \mu) \leqslant L(\tilde{x}, \lambda, \mu)$$

$\tilde{x}$ (Feasible)

$$L(\tilde{x}, \lambda, \mu) = f_o(\tilde{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\tilde{x})}_{\leqslant 0 \text{ for } \lambda \geqslant 0} + \underbrace{\sum_{j=1}^{p} \mu_j h_j(\tilde{x})}_{= 0} \leqslant f_o(\tilde{x})$$

Consequently,

$$\boxed{g(\lambda, \mu) \leq p^*, \quad \lambda \geqslant 0}$$

$*$ $(\lambda, \mu) \in$ dom g with $\lambda \geqslant 0$ (Dual Feasible)

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Example:

Linear Program:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} L(x, \lambda, \mu) &= c^T x - \lambda^T x + \mu^T (Ax - b) \\ &= \mu^T b + (c - \lambda + A^T \mu)^T x \end{aligned}$$

$$g(\lambda, \mu) = \inf_{x \in D} L(x, \lambda, \mu) = \begin{cases} \mu^T b & c - \lambda + A^T \mu = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Summary

Primal Optimization problem

$$\begin{aligned} \text{minimize} \quad & f_o(x) \\ \text{subject to} \quad & f_i(x) \le 0, \quad i = 1, 2, \dots, m \\ & h_j(x) = 0, \quad , j = 1, 2, \dots, p \end{aligned}$$

affine in (λ, μ)

Lagrangian: $L(x, \lambda, \mu) = f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$

concave in (λ, μ)

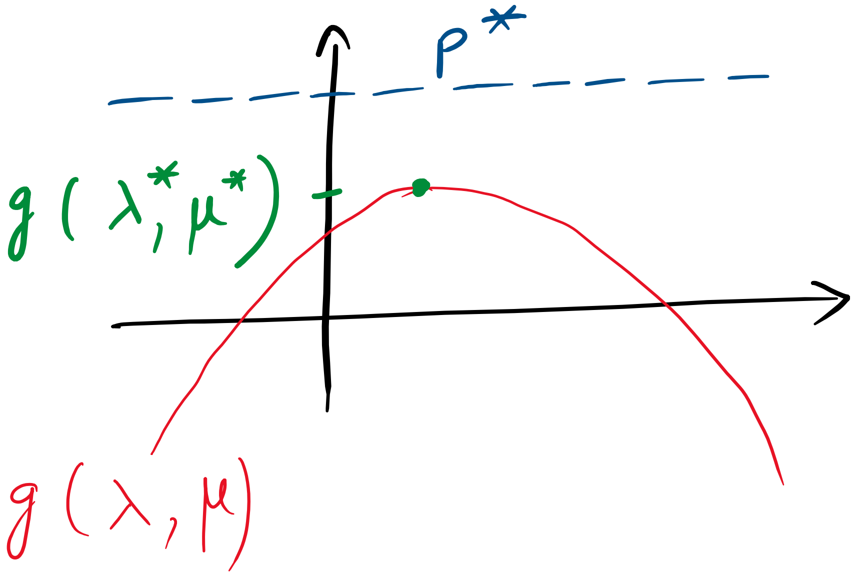
Lagrange Dual Function:

$$g(\lambda, \mu) = \inf_{x \in D} f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

Lower bound on the optimal value of the primal problem:

$$g(\lambda, \mu) \le p^*, \quad \lambda \succeq 0$$

Q: What is the best lower bound?



Feedback: Questions or Comments?

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Slides available at: https://www.zubairkhalid.org/ee563_2020.html
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