Convex Optimization

Duality: Lagrange Dual Problem and Slater's Constraint Qualification

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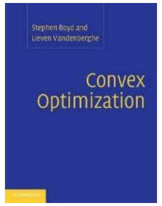
Outline

- Lagrange Dual Problem
- Duality Gap
- Slater’s Condition (constraint qualification)
- Examples



Section 5.2

Summary

Primal Optimization problem

$$\begin{aligned} \text{minimize} \quad & f_o(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & h_j(x) = 0, \quad , j = 1, 2, \ldots, p \end{aligned}$$

affine in (λ, μ)

Lagrangian: $$L(x, \lambda, \mu) = f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

Lagrange Dual Function:

concave in (λ, μ)

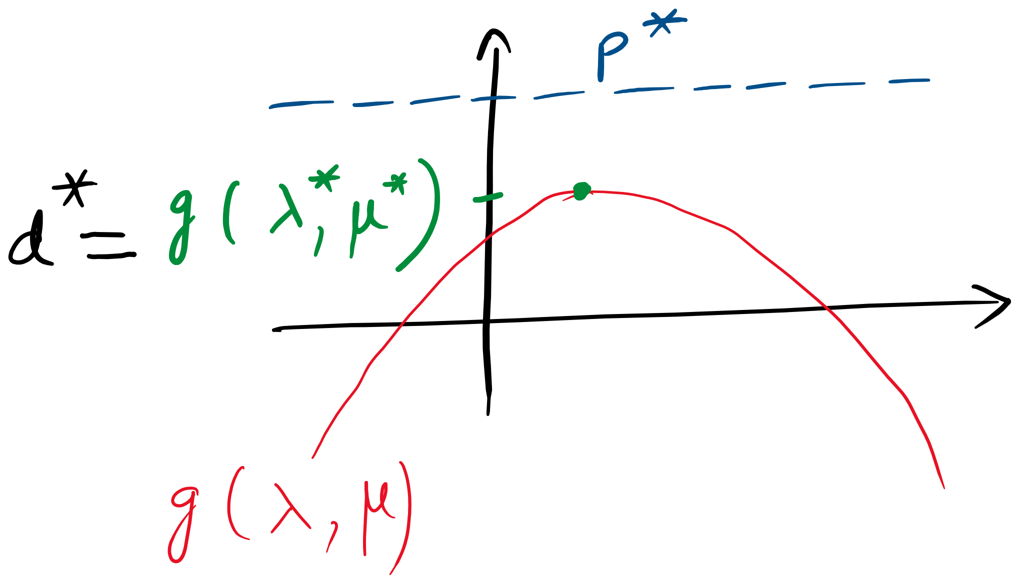
$$g(\lambda, \mu) = \inf_{x \in D} f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

Lower bound on the optimal value of the primal problem:

$$g(\lambda, \mu) \leq p^*, \quad \lambda \succeq 0$$

Q: What is the best lower bound?

Lagrange Dual Problem



Formulation:

$$\begin{aligned} &\text{maximize} \quad g(\lambda, \mu) \\ &\text{subject to} \quad \lambda \succeq 0 \end{aligned}$$

Notes:

* Convex optimization problem
* Feasible set $F = \{(\lambda, \mu) \in \mathbf{dom}\, g \mid \lambda \succeq 0\}$
* (λ^*, μ^*) - Optimal Lagrange multipliers (Dual optimal)

$g(\lambda^*, \mu^*) = d^*$

optimal value of the dual problem

Duality Gap

* Primal problem
  - Convex / Non-convex
  - p^*

Dual Problem
  - Convex
  - d^*

$$d^* \leq p^*$$

$$\text{Duality Gap} \triangleq p^* - d^*$$

If $p^* = d^*$.

$\Rightarrow$ duality gap $= 0$

Strong Duality

If $p^* \geq d^*$

$\Rightarrow$ Weak Duality

Duality Gap

* For primal convex problems, strong duality usually holds

* Constraint Qualifications: conditions under which strong duality holds

* Simple, but powerful condition is referred to as SLATER'S CONDITION OR SLATER'S CONSTRAINT QUALIFICATION.

Slater's Condition (Slater's Theorem)

For convex optimization problem:

$$\begin{aligned} &\text{minimize} && f_o(x) \\ &\text{subject to} && f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & && Ax = b \end{aligned}$$

Strong Duality holds if $\exists\, x \in \text{relint } D$ such that

$Ax = b, \quad f_i(x) < 0, \quad i = 1, 2, \ldots, m.$

* It simply means $\exists$ strictly feasible $x \in \text{relint } D$
* For AFFINE f_i, we only require feasible x, $f_i(x) \leq 0$
* Consequence: For affine constraints, strong duality holds when problem is feasible.

Example: Lagrange Dual Problem for LP

Primal Optimization problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

p^*

Lagrange Dual Function:

$$g(\lambda, \mu) = \begin{cases} \mu^T b & c - \lambda + A^T\mu = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Strong Duality Holds:

$p^* = d^*$

Lagrange Dual Problem:

$$\begin{aligned} \text{maximize} \quad & \mu^T b \\ \text{subject to} \quad & \lambda \geq 0 \\ & A^T\mu + c - \lambda = 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \mu^T b \\ \text{subject to} \quad & A^T\mu + c \geq 0 \end{aligned}$$

d^*.

Example: Least-squares

$A \in R^{p \times n}$

Primal Optimization problem

$$\text{minimize} \quad x^T x$$
$$\text{subject to} \quad Ax = b$$

(Least Norm Signal Estimation)

Lagrangian:

$$\mathcal{L}(x, \mu) = x^T x + \mu^T (Ax - b)$$

Dual Function:

$$g(\mu) = \inf_x \; \mathcal{L}(x, \mu)$$
$$= \inf_x \left(x^T x + \mu^T (Ax - b) \right)$$

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Example: Least-squares

* $\mathcal{L}(x, \mu)$ is convex quadratic

* $\nabla_x \mathcal{L}(x, \mu) = 2x + A^T\mu = 0$

$$\Rightarrow x = -\frac{1}{2} A^T \mu$$

* $g(\mu) = -\frac{1}{4}\mu^T A A^T \mu - b^T\mu$ (Concave)

* $g(\mu) \leq p^*$ (Lower bound)

What is the best lower bound?

Lagrange Dual Problem:

maximize $\quad g(\mu) = -\frac{1}{4}\mu^T A A^T \mu - \mu^T b$

minimize $\quad \frac{1}{4}\mu^T A A^T \mu + \mu^T b$

$$\nabla_\mu g(\mu) = -\frac{1}{4} 2 A A^T \mu - b = 0$$

$$\Rightarrow \mu^* = -2(AA^T)^{-1} b$$

$$d^* = g(\mu^*) = b^T (AA^T)^{-1} b$$

$$\Rightarrow d^* = p^*$$

Primal Problem

minimize $\quad x^T x$

subject to $\quad Ax = b$

$$x^* = A^T (AA^T)^{-1} b$$

$$p^* = (x^*)^T x^*$$

$$= b^T (AA^T)^{-1} b$$

Summary

Primal Optimization problem

$$\begin{aligned} &\text{minimize} \quad f_o(x) \\ &\text{subject to} \; f_i(x) \le 0, \quad i = 1, 2, \ldots, m \\ &\qquad\qquad\quad h_j(x) = 0, \quad, j = 1, 2, \ldots, p \end{aligned}$$

Lagrange Dual Problem:

$$\begin{aligned} &\text{maximize} \quad g(\lambda, \mu) \\ &\text{subject to} \; \lambda \succeq 0 \end{aligned}$$

Best Lower bound on the optimal value of the primal problem:

$$g(\lambda*, \mu*) = d^* \le p^*$$

Duality Gap: $p^* - d^*$

- Weak Duality, $d^* \le p^*$
- Strong Duality, $p^* = d^*$
  - Slater's condition establishes for strong duality

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Feedback: Questions or Comments?

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Slides available at: https://www.zubairkhalid.org/ee563_2020.html
(Let me know should you need latex source)