Convex Optimization

Duality Application: Waterfilling Method for Maximizing Sum Rate of the Communication Channel

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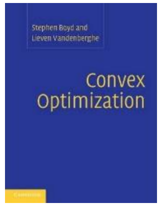
Outline

- Sum-Rate Maximization Problem in Communications
- Solution utilizing KKT Conditions



Section 5.5.3,
Example 5.2

Recap - Karush-Kuhn-Tucker (KKT) Optimality Conditions

Primal Optimization problem

$$\begin{aligned} \text{minimize} \quad & f_o(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & h_j(x) = 0, \quad , j = 1, 2, \ldots, p \end{aligned}$$

Lagrange Dual Problem:

$$\begin{aligned} \text{maximize} \quad & g(\lambda, \mu) \\ \text{subject to} \quad & \lambda \succeq 0 \end{aligned}$$

- Optimal points: x^* and (λ^*, μ^*)

Primal Feasibility

$$f_i(x^*) \leq 0, \quad i = 1, 2, \ldots, m$$

$$h_j(x^*) = 0, \quad j = 1, 2, \ldots, p$$

Dual Feasibility

$$\lambda_i^* \geq 0, \quad i = 1, 2, \ldots, m$$

Complementary Slackness

$$\lambda_i^* \, f_i(x^*) = 0, \quad i = 1, 2, \ldots, m$$

Stationarity

$$\nabla L(x^*, \lambda^*, \mu^*) = \nabla f_o(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) = 0$$

Assumption:

- Strong duality holds (Duality gap is zero)

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Sum Rate Maximization Problem in Communications

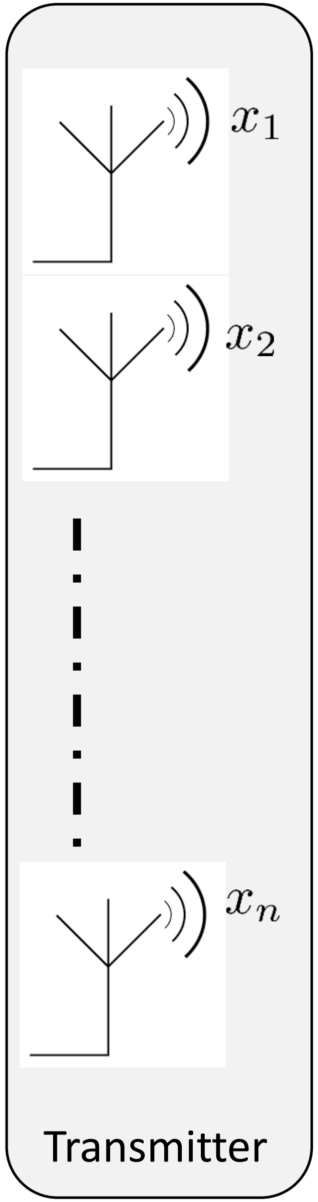
Model:

- MIMO system with n antennas and n channels of equal bandwidth B
- x_i denotes the power allocated to i-th antenna
- g_i denotes the channel power gain assocaited with i-th antenna
- $n_i \sim N(0, \sigma^2)$ denotes the noise across i-th channel
- Total power available is x_T
- Output power across i-th channel is given by

$$y_i = g_i x_i + n_i, \quad i = 1, 2, \ldots, n$$

Problem:

Determine the power allocated to each channel, i.e., $x \in \mathbf{R}^n$, which maximizes the sum-rate (total communication rate or capacity).



Sum Rate Maximization Problem in Communications

Problem Formulation:

* Communication rate or Channel capacity of i-th channel is given by

$$C_i = B \log_2 (1 + SNR_i)$$

* $$SNR_i = \frac{g_i x_i}{\sigma^2}$$

signal power

Noise Power (Expected value)

* $$C_i = B \log_2 \left(1 + \frac{g_i x_i}{\sigma^2}\right)$$

Problem Formulation

* Sum Rate

$$f(x) = \sum_{i=1}^{n} C_i = B \sum_{i=1}^{n} \log_2\left(1 + \frac{g_i x_i}{\delta^2}\right)$$

* $f(x)$ is concave; a sum of concave functions
* Optimization problem can be formulated as ($B = 1, \log_2 \to \log$)

$$\underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \sum_{i=1}^{n} \log\left(1 + \frac{g_i x_i}{\delta^2}\right)$$

Convex Optimization Problem

subject to

$$x \geq 0 \qquad -x \leq 0$$

$$\sum_{i=1}^{n} x_i = x_T \qquad 1^T x = x_T$$

Optimization Problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad -\sum_{i=1}^{n} \log\left(1 + \frac{g_i x_i}{\sigma^2}\right)$$

$$\text{subject to} \quad -x \preceq 0$$

$$\mathbf{1}^T x = x_T$$

Now, we are going to use KKT conditions to solve this problem

- Optimal points: x^* and (λ^*, μ^*) $\lambda^* \in \mathbb{R}^n$, $\mu \in \mathbb{R}$

<u>Primal Feasibility</u>

$$-x^* \preceq 0$$

$$\mathbf{1}^T x^* = x_T$$

<u>Dual Feasibility</u>

$$\lambda^* \succeq 0$$

Optimization Problem

Stationarity

$$\nabla L(x^*, \lambda^*, \mu^*) = 0$$

$$\mathcal{L}(x, \lambda, \mu) = -\sum_{i=1}^{n} \log\left(1 + \frac{g_i x_i}{\sigma^2}\right) + \mu\left(1^T x - x_T\right) - \lambda^T x$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{1}{1 + \frac{g_i x_i}{\sigma^2}} \frac{g_i}{\sigma^2} + \mu - \lambda_i = -\frac{g_i}{\sigma^2 + g_i x_i} + \mu - \lambda_i$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0 \Rightarrow \mu^* = \lambda_i^* + \frac{g_i^*}{\sigma^2 + g_i x_i^*}, \quad i = 1, 2, \dots, n$$

Optimization Problem

Complementary Slackness $\lambda_i^* x_i^* = 0$

Case 1

$\lambda_i^* = 0, \quad x_i^* > 0$

$$\Rightarrow \mu^* = \frac{g_i}{\sigma^2 + g_i x_i^*}$$

$$\Rightarrow \frac{1}{\mu^*} = x_i^* + \frac{\sigma^2}{g_i}$$

$$\Rightarrow \frac{1}{\mu^*} > \frac{\sigma^2}{g_i}$$

$$x^* = \frac{1}{\mu^*} - \frac{\sigma^2}{g_i}$$

Case 2

$x_i^* = 0 \Rightarrow \lambda_i^* > 0$

$$\mu^* = \frac{g_i}{\sigma^2} + \lambda_i^*$$

$$\Rightarrow \mu^* > \frac{g_i}{\sigma^2}$$

$$\frac{1}{\mu^*} < \frac{\sigma^2}{g_i}$$

$$x_i^* = \begin{cases} \frac{1}{\mu^*} - \frac{\sigma^2}{g_i} & \frac{1}{\mu^*} \geq \frac{\sigma^2}{g_i} \\ 0 & \frac{1}{\mu^*} < \frac{\sigma^2}{g_i} \end{cases}$$

$$x_i^* = \max\left(\frac{1}{\mu^*} - \frac{\sigma^2}{g_i}, 0\right)$$

Optimization Problem

* To find μ^*, we note that x^* is primal feasible, i.e.,

$$\mathbf{1}^T x^* = x_T \implies \sum_{i=1}^{n} x_i^* = x_T$$

$$\boxed{\sum_{i=1}^{n} \max\left(\frac{1}{\mu^*} - \frac{\sigma^2}{g_i}, 0\right) = x_T}$$

water-filling Method

$$\boxed{x_i^* = \max\left(\frac{1}{\mu^*} - \frac{\sigma^2}{g_i}, 0\right)}$$

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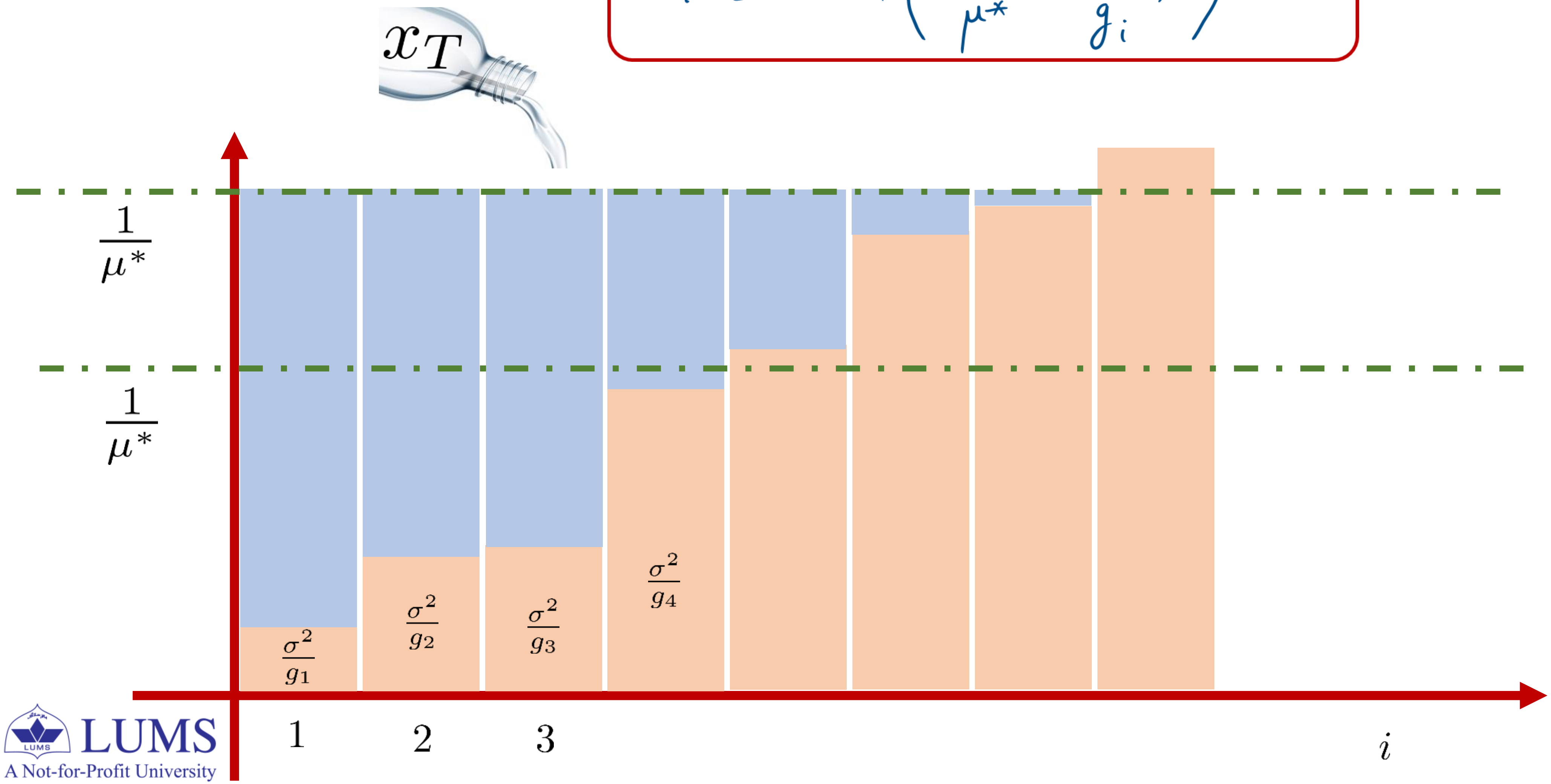
Water-filling Interpretation

- Assume $g_1 \geq g_2 \geq g_3 \ldots$ (without loss of generality)

$$\boxed{x_i^* = \max\left(\frac{1}{\mu^*} - \frac{\sigma^2}{g_i}, 0\right)}$$



Feedback: Questions or Comments?

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Slides available at: https://www.zubairkhalid.org/ee563_2020.html

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