

BAND-LIMITED EXTRAPOLATION ON THE SPHERE FOR SIGNAL RECONSTRUCTION IN THE PRESENCE OF NOISE

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ABSTRACT

We investigate the problem of extrapolation of band-limited signals on the 2-sphere in the presence of noise. Specifically, given incomplete or spatially limited measurements subject to noise, find the unique extrapolation to the complete 2-sphere. We present an analytic solution to the extrapolation problem based on the expansion of a signal in Slepian basis corresponding to an orthogonal set of eigenfunctions of an associated energy concentration problem. An alternative equivalent iterative algorithm is also developed for practical implementation and guidelines are proposed to choose the parameters of the iterative algorithm. The capability of the proposed extrapolation is compared and demonstrated with the help of an illustration example.

Index Terms— 2-sphere; signal extrapolation; bandlimited signals; spherical harmonics.

1. INTRODUCTION

The development of signal processing techniques for signals defined on the sphere finds applications in various branches of sciences and engineering (e.g., [1–5]). In this paper, we consider the fundamental signal processing problem of extrapolation of a band-limited signal from its noisy observations taken over a limited or incomplete spatial region on the sphere. Using the Earth topography, as an example, a region can be irregular such as a continent or regular such as a polar cap region, without any essential change in the theory which underpins concentration [3] and extrapolation problems [5].

1.1. Relation to Prior Work

The problem of band-limited signal extrapolation has been extensively studied for signals in the time domain [6–9]. The iterative methods to extrapolate the signal, based on the successive reduction of the mean-square error are generally preferred as compared to the analytic solutions due to the difficulty in implementation of the analytic methods [8, 9]. If the observations are subject to noise, the extrapolation problem becomes ill-conditioned and the regularized solutions are often considered [9], which requires additional knowledge about the signal beyond its band-limited character.

For signals on the sphere, the Papoulis algorithm has been revisited and various algorithms have been proposed [5, 10, 11]. An analogue of the Papoulis algorithm, which exploits the bandlimiting characteristic of a signal to be extrapolated, is presented in [5, 10]. In the discrete (sampled) spatial domain, an iterative gradient algorithm which converges to the minimum norm least square solution

is proposed in [12] and its modified version with fast convergence has been developed in [11]. The regularized solution is developed to counter the effect of noise in the extrapolation of signal [12]. However, the choice of the regularization parameter is not considered.

1.2. Contributions

In this work, we consider the problem of extrapolating a signal from its noisy measurements made over the limited spatial region. We present an analytic solution based on the expansion of a signal in Slepian basis, obtained as a solution of concentration problem on the sphere [3]. The proposed solution takes into account the information about the bound on the noise energy in the region over which the observations are made. We also present an equivalent iterative solution which does not require the computation of Slepian basis and converges to the analytic solution. The implementation of the proposed method is also outlined and illustration is provided to show the capability of the proposed iterative algorithm.

2. PRELIMINARIES

2.1. Signals on the 2-Sphere

We consider the complex Hilbert space finite energy functions on the 2-sphere, $L^2(\mathbb{S}^2)$, equipped with inner product for functions f, g given by

$$\langle f, g \rangle \triangleq \int_{\mathbb{S}^2} f(\hat{\mathbf{x}}) \overline{g(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}), \quad (1)$$

which induces a norm $\|f\| \triangleq \langle f, f \rangle^{1/2}$. Here, $\overline{(\cdot)}$ denotes the complex conjugate operation and $\hat{\mathbf{x}} \triangleq (\sin \theta \cos \theta, \sin \theta \sin \phi, \cos \theta)^T \in \mathbb{S}^2 \subset \mathbb{R}^3$ stands for a point on the 2-sphere, where $(\cdot)^T$ represents the vector transpose, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ denote the co-latitude and longitude respectively and $ds(\hat{\mathbf{x}}) = \sin \theta d\theta d\phi$ is the surface measure on the 2-sphere. Also define $\langle f, g \rangle_R \triangleq \int_R f(\hat{\mathbf{x}}) \overline{g(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}})$ as the inner product of functions evaluated over the region $R \subset \mathbb{S}^2$ and $\|f\|_R \triangleq \langle f, f \rangle_R^{1/2}$ as the energy of the signal f with in the region R . The functions with finite energy (induced norm) are referred as signals on the sphere.

2.2. Spherical Harmonics

Spherical harmonic functions (or spherical harmonics for short), denoted by $Y_\ell^m(\hat{\mathbf{x}}) = Y_\ell^m(\theta, \phi)$ [13], defined for integer degree $\ell \geq 0$ and integer order $m \leq |\ell|$ serve as complete orthonormal set of basis functions. Therefore, a signal $f \in L^2(\mathbb{S}^2)$ can be expressed as

$$f(\hat{\mathbf{x}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_\ell^m Y_\ell^m(\hat{\mathbf{x}}), \quad (2)$$

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where

$$(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle \quad (3)$$

denotes the spherical harmonic coefficient of degree ℓ and order m and form the spectral representation of signal. The signal f is said to be band-limited at degree L if $(f)_\ell^m = 0$ for $\ell > L$. The set of such band-limited signals forms a subspace of $L^2(\mathbb{S}^2)$ and is denoted by \mathcal{H}_L .

2.3. Operators on the Sphere

Define an operator \mathcal{K} for signals on the sphere using general Fredholm integral equation

$$(\mathcal{K}f)(\hat{\mathbf{x}}) = \int_{\mathbb{S}^2} K(\hat{\mathbf{x}}, \hat{\mathbf{y}}) f(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}), \quad (4)$$

where $K(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is the kernel for an operator \mathcal{K} . Using the definition in (4)

Definition 1 (Spatial Selection Operator). *Define the spatial selection operator \mathcal{K}_R that selects the function in a region $R \subset \mathbb{S}^2$ with kernel given by*

$$K_R(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \triangleq I_R(\hat{\mathbf{x}})\delta(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \quad (5)$$

where $I_R(\hat{\mathbf{x}}) = 1$ for $\hat{\mathbf{x}} \in R \subset \mathbb{S}^2$ and $I_R(\hat{\mathbf{x}}) = 0$ for $\hat{\mathbf{x}} \in \mathbb{S}^2 \setminus R$ is an indicator function of the region R and $\delta(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ denotes the Dirac delta function on the sphere.

Definition 2 (Spectral Selection Operator). *Define the spectral selection operator \mathcal{K}_L which band-limits the signal with maximum spherical harmonic degree L with kernel given by*

$$K_L(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \triangleq \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{\mathbf{x}}) \overline{Y_\ell^m(\hat{\mathbf{y}})}. \quad (6)$$

Since both the operators are projections operator, they are idempotent. Furthermore, they are also self-adjoint in nature.

2.4. Slepian Concentration Problem on the Sphere

Analogous to the Slepian concentration problem in time and frequency domain, the concentration problem on the sphere for finding the functions with simultaneous concentration in both spatial and spectral domains has been studied [3, 14]. In order to maximize the spatial concentration of a bandlimited signal $h \in L^2(\mathbb{S}^2)$ with band-limit L within the region R , the spatial concentration ratio

$$\lambda = \frac{\|h\|_R^2}{\|h\|^2} \quad (7)$$

is maximized, where $0 < \lambda < 1$ is a measure of spatial concentration. The concentration problem in (7) can be expressed in spectral domain as

$$\lambda = \frac{\sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} \sum_{\ell'=0}^L \sum_{m'=-\ell'}^{\ell'} (h)_\ell^m \overline{(h)_{\ell'}^{m'}} \mathcal{E}_{\ell\ell', mm'}}{\sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} |(h)_\ell^m|^2}, \quad (8)$$

where $|\cdot|$ gives the magnitude and

$$\mathcal{E}_{\ell\ell', mm'} = \int_R Y_\ell^m(\hat{\mathbf{x}}) \overline{Y_{\ell'}^{m'}(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}). \quad (9)$$

Provided $\mathcal{E}_{\ell\ell', mm'}$ can be computed over the region R , the concentration problem in (8) can be solved as an algebraic eigenvalue problem, the solution of which gives $(L+1)^2$ orthonormal eigenfunctions. Let the eigenfunctions be denoted by h_p , $p \in [1, 2, \dots, (L+1)^2]$ and the associated eigenvalue for each eigenfunction is given by λ_p . We note that the eigenfunctions serve as an alternative *complete* basis, referred as Slepian basis, for the representation of a signal on the sphere. Furthermore, the eigenfunctions are orthogonal over the region R , that is,

$$\langle h_p, h_q \rangle = \delta_{p,q}, \quad \langle h_p, h_q \rangle_R = \lambda_p \delta_{p,q}. \quad (10)$$

3. PROBLEM STATEMENT

Let $f \in \mathcal{H}_L$ be the band-limited signal with maximum degree L . Furthermore, assume that f is only known or can be only observed over some region $R \subset \mathbb{S}^2$ on the sphere, where the observations are also subject to noise. Let the known signal be denoted by $g \in L^2(\mathbb{S}^2)$, given by

$$g(\hat{\mathbf{x}}) = (\mathcal{K}_R f)(\hat{\mathbf{x}}) + z(\hat{\mathbf{x}}), \quad (11)$$

where z denotes the noise.

Given $g(\hat{\mathbf{x}}) \in L^2(\mathbb{S}^2)$, we consider the problem of determining $\tilde{f}(\hat{\mathbf{x}}) \in \mathcal{H}_L$ as an estimate of the signal $f(\hat{\mathbf{x}}) \in \mathcal{H}_L$ for all $\hat{\mathbf{x}} \in \mathbb{S}^2$ with the assumption that the energy of the noise in the region R is less than or equal to the known bound ϵ^2 , that is, $\|z\|_R^2 \leq \epsilon^2$. Mathematically, find $\tilde{f}(\hat{\mathbf{x}}) \in \mathcal{H}_L$ such that its norm is minimized, that is,

$$\min \|\tilde{f}\|^2, \quad (12)$$

under the constraint

$$\|\tilde{f} - g\|_R^2 \leq \epsilon^2. \quad (13)$$

4. PROPOSED EXTRAPOLATION SOLUTIONS

Here we present the solution to the signal extrapolation problem presented in previous section. The result is presented in the form of the following theorem.

Theorem 1. *The band-limited extrapolated signal $\tilde{f}(\hat{\mathbf{x}}) \in \mathcal{H}_L$ satisfying the minimization condition in (12) with respect to the constraint in (13) can be determined from the noisy observations $g(\hat{\mathbf{x}})$ given in (11) as*

$$\tilde{f}(\hat{\mathbf{x}}) = \sum_{p=1}^{(L+1)^2} a_p h_p(\hat{\mathbf{x}}), \quad (14)$$

with

$$a_p = \frac{\mu \lambda_p b_p}{1 + \mu \lambda_p}, \quad (15)$$

where

$$b_p \triangleq \frac{1}{\lambda_p} \langle g, h_p \rangle_R, \quad (16)$$

and $\mu > 0$ and is chosen as a solution of

$$\sum_{p=1}^{(L+1)^2} \frac{\lambda_p |b_p|^2}{(\mu \lambda_p + 1)^2} = \epsilon^2. \quad (17)$$

Proof. By defining $a_p \triangleq \langle \tilde{f}, h_p \rangle$ and using (16), we write \tilde{f} and g as

$$\tilde{f}(\hat{\mathbf{x}}) = \sum_{p=1}^{(L+1)^2} a_p h_p(\hat{\mathbf{x}}), \quad (18)$$

$$g(\hat{\mathbf{x}}) = \sum_{p=0}^{(L+1)^2} b_p h_p(\hat{\mathbf{x}}), \quad \hat{\mathbf{x}} \in R, \quad (19)$$

and formulate the objective function in (12), which is required to be minimized, as

$$O = \left(\sum_{p=1}^{(L+1)^2} |a_p|^2 \right), \quad (20)$$

and the constraint in (13)

$$\sum_{p=1}^{(L+1)^2} \lambda_p |b_p - a_p|^2 \leq \epsilon^2. \quad (21)$$

Formulating the minimization problem in (12) using the Lagrangian multiplier as follows

$$\mathcal{L} = \sum_{p=1}^{(L+1)^2} |a_p|^2 + \mu \lambda_p |b_p - a_p|^2, \quad (22)$$

where μ is the Lagrangian multiplier. Taking the derivative with respect to a_p and equating the result to 0 gives

$$a_p + \mu \lambda_p a_p - \mu \lambda_p b_p = 0, \quad (23)$$

which when solved for a_p yields the result stated in (15).

Combining equations (15) and (21), μ can be determined as

$$M(\mu) \triangleq \sum_{p=1}^{(L+1)^2} \frac{\lambda_p |b_p|^2}{(\mu \lambda_p + 1)^2} \leq \epsilon^2, \quad (24)$$

such that it minimizes the objective function in (20) for a_p given in (15). Since $\lambda_p > 0$, we note that $M(-\mu) > M(\mu)$, which implies $M(\mu)$ is monotonically decreasing function of μ and $\mu > 0$ should be chosen. However, the objective function in (20) is minimized for smaller value of μ . Combining this fact with the requirement in (21), μ should be the positive root of (17) if it exists or otherwise $\mu = \infty$. \square

Corollary 1. For any value of μ , it can be shown that

$$\|f\|_R^2 + \frac{2}{\mu} O + M(\mu) = \|g\|_R^2. \quad (25)$$

4.1. Iterative Algorithm

The implementation of the solution to the extrapolation problem presented in Theorem 1 requires the computation of Slepian basis for given L and the region $R \subset \mathbb{S}^2$. The Slepian basis can be determined for azimuthally symmetric regions [3], however these cannot be exactly computed for arbitrary region R . Here, we address this problem and propose an equivalent iterative algorithm in the form of the following theorem.

Theorem 2. An iterative method, equivalent to the solution proposed in Theorem 1, to determine a band-limited extrapolated signal $\tilde{f}(\hat{\mathbf{x}}) \in L^2(\mathbb{S}^2)$ from noisy measurements $g(\hat{\mathbf{x}})$ given in (11) is

$$\tilde{f}_{n+1}(\hat{\mathbf{x}}) = \mathcal{K}_L \left[(1 - \alpha/\mu) \tilde{f}_n(\hat{\mathbf{x}}) + \alpha \mathcal{K}_R \left(g(\hat{\mathbf{x}}) - \tilde{f}_n(\hat{\mathbf{x}}) \right) \right], \quad (26)$$

where $\tilde{f}_n(\hat{\mathbf{x}})$ and $\tilde{f}_{n+1}(\hat{\mathbf{x}})$ denote the extrapolated signal at the n^{th} and $(n+1)^{\text{th}}$ iterations respectively and the initial condition is $f_0(\hat{\mathbf{x}}) = 0$. Here choose

$$0 < \alpha < \frac{2\mu}{1 + \mu}. \quad (27)$$

and

$$\mu > \frac{\|g(\hat{\mathbf{x}})\|_R - \epsilon}{\epsilon}. \quad (28)$$

Proof. We show that the iterative solution presented (26) converges to the solution presented in Theorem 1, that is,

$$\lim_{n \rightarrow \infty} \tilde{f}_{n+1}(\hat{\mathbf{x}}) = \tilde{f}(\hat{\mathbf{x}}). \quad (29)$$

By defining

$$\tilde{f}_n(\hat{\mathbf{x}}) \triangleq \sum_{p=1}^{(L+1)^2} a_{p,n} h_p(\hat{\mathbf{x}}), \quad (30)$$

and using (26) and employing the orthogonality of Slepian basis, we can determine the coefficients $a_{p,n+1}$ as

$$a_{p,n+1} = (1 - \alpha/\mu - \alpha \lambda_p) a_{p,n} + \alpha \lambda_p b_p, \quad (31)$$

or recursively by putting $a_{p,0} = 0$ as

$$a_{p,n} = \frac{\lambda_p b_p}{\lambda_p + 1/\mu} [1 - (1 - \alpha(\lambda_p + 1/\mu))^n]. \quad (32)$$

Taking a value for α as defined in (27), we obtain

$$\lim_{n \rightarrow \infty} |a_p - a_{p,n}|^2 = 0, \quad (33)$$

which is equivalent to (29). For the value of μ , following the result of Corollary 1 in (25) gives

$$\frac{\mu \|g\|_R^2 - \mu M(\mu)}{\mu + 2} > O > \mu^2 M(\mu), \quad (34)$$

or equivalently

$$\mu > \frac{\|g\|_R - \sqrt{M(\mu)}}{\sqrt{M(\mu)}},$$

which when compared to (17) gives (28) and completes the proof of theorem. \square

Remark 1. We note that the proposed iterative algorithm does not require the computation of eigenfunctions and only requires the information about the bound on the noise level. If the observed signal is not corrupted with noise, that is, $\epsilon = 0$, which implies $\mu = \infty$, we note that the iterative algorithm in (26) becomes the iterative algorithm for extrapolation proposed in [5, 10].

4.2. Implementation

In practice, the region $\mathbb{S}^2 \setminus R$ of the sphere over which the observations are not available or to be extrapolated can be a union of different non-connected regions. For example, the estimation of the unobserved gravity data near the poles is well known in geodesy [15]. A similar problem exists in the estimation of head-related transfer function [16] and has been considered in [5] in the development of extrapolation algorithm. We present the implementation of an iterative algorithm presented in Theorem 2 in the most general setting, where the observations are made randomly over the sphere and do not need to be in some connected region.

We use equiangular sampling scheme proposed in [17] which requires $N = 2(L+1)^2$ number of samples to represent band-limited signal on the sphere. Let $\hat{\mathbf{w}} = [\hat{w}_1, \hat{w}_2, \dots, \hat{w}_N]$ denote the N samples on the sphere, which are ordered such that the observations are available over the first $Q \leq N$ samples. We define the signal \mathbf{f} in sampled spatial domain as

$$\mathbf{f} \triangleq [f(\hat{w}_1), f(\hat{w}_2), \dots, f(\hat{w}_N)]^T \quad (35)$$

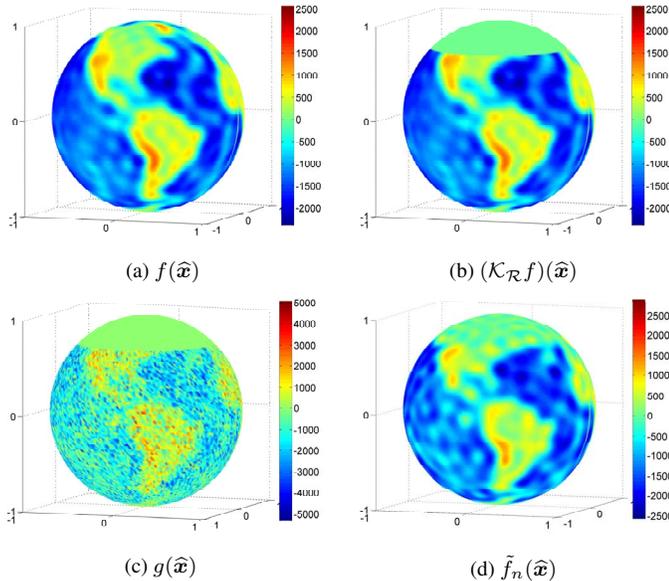


Fig. 1: (a) The Earth topographic map as signal $f(\hat{\mathbf{x}})$ with band-limit $L - 31$ (b) Spatially limited signal $(\mathcal{K}_R f)(\hat{\mathbf{x}})$, (c) the known signal $(\mathcal{K}_R f)(\hat{\mathbf{x}}) + z(\hat{\mathbf{x}})$ and (d) the extrapolated signal $\tilde{f}_n(\hat{\mathbf{x}})$ for $n = 50$.

and known noise corrupted signal \mathbf{g} as

$$\mathbf{g} \triangleq [g(\hat{w}_1), g(\hat{w}_2), \dots, g(\hat{w}_Q)]^T. \quad (36)$$

The iterative algorithm requires the implementation of spatial selection and spectral selection operators. The spatial selection operation for Q number of sample points over which the signal is known can be implemented in the form of matrix $\mathbf{D} = \{D_{u,v}\}$ of size $Q \times N$ with entries given by

$$D_{u,v} = \begin{cases} 1 & 0 \leq u = v \leq Q \\ 0 & \text{otherwise} \end{cases} \quad (37)$$

which selects the first Q samples of a signal consisting of N samples. Note that the matrix operator \mathbf{D}^T appends $N - Q$ zeros to the Q sample measured signal. The spectral selection operation to band-limit the signal up to degree L is performed by taking the spherical harmonic transform of \mathbf{f} or $\mathbf{D}^T \mathbf{g}$.

5. ILLUSTRATION

Here, we illustrate the capability of the proposed algorithm to extrapolate the signal in a noisy environment. We consider the Earth topographic map as a signal $f(\hat{\mathbf{x}})$ shown in Fig. 1 (a), band-limited to $L = 31$, which is synthesized using the spherical harmonic model of the topography of Earth. In order to quantify the quality of any signal $h(\hat{\mathbf{x}})$ with respect to the original (reference) signal $f(\hat{\mathbf{x}})$, we define the signal-to-noise ratio (SNR) for the signal $h(\hat{\mathbf{x}})$ as

$$\text{SNR}^h = 20 \log \frac{\|f\|}{\|f - h\|}.$$

We choose $R = \{\pi/4 \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}$ over which the measurements are taken. The spatially limited signal $\mathcal{K}_R f$ is shown in

¹Spherical harmonic model of Earth topographic map is available at <http://www.ipgp.fr/~wieczor/SH/>.

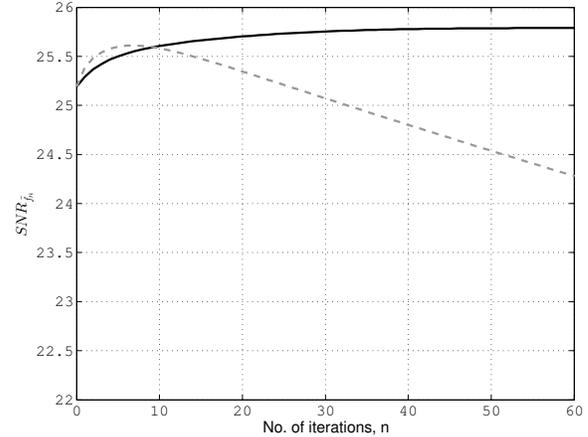


Fig. 2: $\text{SNR}^{\tilde{f}_n}$ is plotted against number of iterations for the extrapolated signal \tilde{f}_n using the proposed iterative method (solid line) and the algorithm reported in [5] (dashed).

Fig. 1(b) with $\text{SNR}^{\mathcal{K}_R f} = 28.96\text{dB}$. We assume that the measurements are subject to additive Gaussian noise and the known noise corrupted signal is shown in Fig. 1(c) with $\text{SNR}^g = 6.87\text{dB}$. Using the proposed iterative algorithm with $\mu = 22$ and $\alpha = 0.34$, the extrapolated signal \tilde{f}_n is obtained with $\text{SNR}^{\tilde{f}_n} = 25.76\text{dB}$ and is shown in Fig. 1(d) for $n = 50$.

We also compare the improvement of our algorithm with an iterative gradient algorithm proposed in [5]. $\text{SNR}^{\tilde{f}_n}$ of the extrapolated signal is compared in Fig. 2 against the number of iterations. Since the method in [5] does not take into account the presence of noise in the signal, its solution deviates from the original signal with the number of iterations.

6. CONCLUDING REMARKS

We have considered the extrapolation of band-limited signals defined on the 2-sphere. Taking into account the bound on the energy of the noise corrupting the measurements made over incomplete or limited spatial domain, an analytic solution based on the expansion of signal in the Slepian basis has been presented for extrapolation to the complete sphere. An alternative iterative solution has been developed for practical implementation, where we have also devised bounds on the parameters of the algorithm.

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