MINIMUM MEAN SQUARE ERROR EQUALIZATION ON THE 2-SPHERE

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ABSTRACT

In this paper we consider the zero-forcing (ZF) and minimum mean square error (MMSE) criteria for signal recovery using linear operators as equalizers for signals observed on the 2-sphere that are subject to linear distortions and noise. The distortions considered are bounded operators and can include convolutions, rotations, spatial and spectral truncations, projections or combinations of these. Likewise the signal and noise are very general being modeled as anisotropic stochastic processes on the 2-sphere. In both the distortion model and signal model the findings in this paper are significantly more general than results that can be found in the literature. The MMSE equalizer is shown to reduce to the ZF equalizer when the distortion operator has an inverse and there is an absence of noise. The ability of the MMSE to recover a Mars topography map signal from a projection operator, which fails to have a ZF solution, is given as an illustration.

***Index Terms*—** 2-sphere; unit sphere; equalization; MMSE; zero-forcing.

1. INTRODUCTION

The development of signal processing techniques for signals defined on the 2-sphere finds many applications in various fields of science and engineering. These applications include surface analysis in medical imaging [1], gravity modelling in geophysics [2], sound reproduction in acoustics [3] and wireless channel modeling [4] in communication systems.

Signals observed on the 2-sphere are often subject to linear distortions and noise [2, 5] and hence, signal recovery using appropriate equalization is required. In this paper we consider the zero-forcing (ZF) and minimum mean square error (MMSE) criteria for signal recovery using linear operators as equalizers.

1.1. Relation to Prior Work

Existing results in the literature include [6] which considered ZF and MMSE equalization for a special class of signals and operations, namely for isotropic input signal and noise and for isotropic convolution. In this case, the operator matrix of isotropic convolution and the equalizer are both diagonal with a certain structure for the diagonal elements. Deconvolution problems for recovering cosmic microwave background (CMB) signal have been considered in [5,7], again with an assumption of isotropic noise. Furthermore, spectral weighting, of which isotropic convolution is an example [8], has been considered for smoothing in medical imaging [1]. However, in reality, the input signal and noise may be non-isotorpic having gone through a general system with a non-diagonal operator matrix. The general form of ZF and MMSE equalizers for signals on the 2-sphere has not been studied or applied before.

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In this paper, we fill this gap and provide the general form of ZF and MMSE equalizers for signals on the 2-sphere. This can serve as a reference for numerous applications of equalization on the 2-sphere as discussed above. We then specialize the formulation to important classes of distortion operators such as spectral truncation, rotation, projection into the subspace of azimuthally symmetric signals and spectral weighting. We finally illustrate the capability of the proposed equalization framework for signal recovery through an example.

2. SIGNALS AND SYSTEMS ON THE 2-SPHERE

2.1. 2-Sphere

The 2-sphere is denoted by $\mathbb{S}^2$ and defined as $\mathbb{S}^2 \triangleq \{\boldsymbol{u} \in \mathbb{R}^3 \colon \|\boldsymbol{u}\| = 1\}$. A representative unit vector in $\mathbb{S}^2$ is denoted by $\widehat{\boldsymbol{u}} \equiv \widehat{\boldsymbol{u}}(\theta,\phi) \triangleq (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)'$, where $(\cdot)'$ is vector transpose. $\theta \in [0,\pi]$ is the co-latitude with respect to the positive $z-$axis and $\phi \in [0, 2\pi)$ is the longitude with respect to the positive $x-$axis in the $x-y$ plane.

2.2. Signals on 2-Sphere

The space of complex-valued square-integrable functions such as f and h, whose domain is $\mathbb{S}^2$, is a Hilbert space denoted by $L^2(\mathbb{S}^2)$ with the inner product defined as

$$\langle f, h\rangle \triangleq \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{u}})\overline{h(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}), \tag{1}$$

with the induced norm $\|f\| \triangleq \langle f, f\rangle^{1/2}$. Also, $ds(\widehat{\boldsymbol{u}}) = \sin\theta\, d\theta\, d\phi$ is the differential area element. A finite energy function in $L^2(\mathbb{S}^2)$ with $\|f\| < \infty$ is referred to as a "signal on the 2-sphere" or simply a "signal".

2.3. Spherical Harmonics and Spectral (Fourier) Domain

The spherical harmonic function $Y_\ell^m(\theta,\phi)$ for degree $\ell \geq 0$ and order $|m| \leq \ell$ is defined as [9]

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)e^{im\phi}, \tag{2}$$

where P_ℓ^m is the associated Legendre function of degree ℓ and order m [9]. Spherical harmonic functions form a complete basis for $L^2(\mathbb{S}^2)$. By their completeness, any signal $f \in L^2(\mathbb{S}^2)$ can be expanded as

$$f(\widehat{\boldsymbol{u}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (f)_\ell^m Y_\ell^m(\widehat{\boldsymbol{u}}), \tag{3}$$

where equality is understood in the sense of convergence in the mean. In the sequel, we will use the shorthand notations of the form $\sum_{\ell,m}$ instead of $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}$. And where

$$(f)_\ell^m \triangleq \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{u}})\overline{Y_\ell^m(\widehat{\boldsymbol{u}})}\, ds(\widehat{\boldsymbol{u}}) \tag{4}$$

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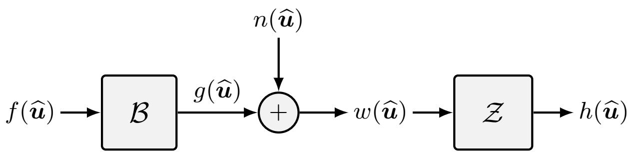


Fig. 1: System diagram with bounded distortion operator $\mathcal{B}$ and operator $\mathcal{Z}$ representing an equalizer. $f(\widehat{\boldsymbol{u}})$ is the desired signal, $g(\widehat{\boldsymbol{u}})$ is the distorted signal, $n(\widehat{\boldsymbol{u}})$ is the noise, $w(\widehat{\boldsymbol{u}})$ is the noisy distorted signal, and $h(\widehat{\boldsymbol{u}})$ is the equalized output.

is the spherical harmonic Fourier coefficient of degree ℓ and order m. The infinite-dimensional spectral (column) vector containing all such $(f)_\ell^m$ is denoted by $\mathbf{f}$, ordered by convention as $\mathbf{f} = \big((f)_0^0, (f)_1^{-1}, (f)_1^0, (f)_1^1, (f)_2^{-2}, \cdots\big)'$.

We use $\delta_{\ell,p}$ to denote the Kronecker delta function, which is equal to one only when $\ell = p$ and is zero otherwise. Also define $\delta_{\ell,p}^{m,q} \triangleq \delta_{\ell,p}\delta_{m,q}$.

2.4. Stochastic Processes on 2-Sphere

For a stochastic signal $f \in L^2(\mathbb{S}^2)$, the cross correlation between its spherical harmonic Fourier coefficients is

$$F_{\ell,p}^{m,q} \triangleq \mathbb{E}\{(f)_\ell^m \overline{(f)_p^q}\}, \tag{5}$$

where $\mathbb{E}\{\cdot\}$ denotes statistical expectation. The covariance matrix containing all $F_{\ell,p}^{m,q}$ is

$$\mathbf{F} \triangleq \mathbb{E}\{\mathbf{f}\mathbf{f}^H\}, \tag{6}$$

where the superscript H denotes Hermitian operation and the same ordering as in $\mathbf{f}$ is used for its columns and rows. For an isotropic stochastic signal [10], $F_{\ell,p}^{m,q}$ reduces to

$$F_{\ell,p}^{m,q} = F_\ell \delta_{\ell,p}^{m,q}$$

resulting in a diagonal covariance matrix $\mathbf{F}$ with identical diagonal elements F_ℓ for a given degree ℓ and all orders $|m| \le \ell$.

2.5. Bounded Operators and their Spectral Representation

A bounded linear operator $\mathcal{B}$ with kernel $B(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}})$, where $\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}} \in \mathbb{S}^2$, maps the input signal $f \in L^2(\mathbb{S}^2)$ to the output signal $g \in L^2(\mathbb{S}^2)$ as

$$g(\widehat{\boldsymbol{u}}) = (\mathcal{B}f)(\widehat{\boldsymbol{u}}) = \int_{\mathbb{S}^2} B(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}) f(\widehat{\boldsymbol{v}}) ds(\widehat{\boldsymbol{v}}). \tag{7}$$

By defining

$$b_{\ell,p}^{m,q} \triangleq \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} B(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}) \overline{Y_\ell^m(\widehat{\boldsymbol{u}})} Y_p^q(\widehat{\boldsymbol{v}})\, ds(\widehat{\boldsymbol{u}})\, ds(\widehat{\boldsymbol{v}}), \tag{8}$$

we can write the operator action given in (7) in spectral domain as

$$(g)_\ell^m = \sum_{p,q} b_{\ell,p}^{m,q} (f)_p^q, \tag{9}$$

or equivalently express the vector of Fourier coefficients of the output signal $\mathbf{g}$ as

$$\mathbf{g} = \mathbf{B}\mathbf{f}, \tag{10}$$

where $\mathbf{B}$ is the operator matrix (in the spherical harmonic basis) for operator $\mathcal{B}$ with entries $b_{\ell,p}^{m,q}$ and the same ordering as in $\mathbf{f}$ is used for indexing of rows (identified by ℓ, m) and columns (identified by p, q). For the case when input signal is a stochastic process with covariance matrix $\mathbf{F}$, the output g is characterized by its covariance matrix $\mathbf{G}$ given by

$$\mathbf{G} = \mathbf{B}\mathbf{F}\mathbf{B}^H. \tag{11}$$

3. EQUALIZATION

3.1. Problem Formulation

The block diagram of the system under consideration is shown in Fig. 1. The desired signal $f(\widehat{\boldsymbol{u}})$ can be only observed after going through the system with the bounded linear operator or distortion $\mathcal{B}$ and contamination with additive noise $n(\widehat{\boldsymbol{u}})$ resulting in

$$w(\widehat{\boldsymbol{u}}) = g(\widehat{\boldsymbol{u}}) + n(\widehat{\boldsymbol{u}}) = \big(\mathcal{B}f\big)(\widehat{\boldsymbol{u}}) + n(\widehat{\boldsymbol{u}}). \tag{12}$$

Signal $w(\widehat{\boldsymbol{u}})$ then goes through our designed equalization operator $\mathcal{Z}$ resulting in output signal

$$h(\widehat{\boldsymbol{u}}) = \big(\mathcal{Z}w\big)(\widehat{\boldsymbol{u}}). \tag{13}$$

Using the matrix representation of the operators and spectral representation of signals and noise, we have the following relations:

$$\mathbf{w} = \mathbf{B}\mathbf{f} + \mathbf{n}, \tag{14}$$

$$\mathbf{h} = \mathbf{Z}\mathbf{w} = \mathbf{Z}\mathbf{B}\mathbf{f} + \mathbf{Z}\mathbf{n}, \tag{15}$$

and covariance

$$\mathbf{W} = \mathbf{B}\mathbf{F}\mathbf{B}^H + \mathbf{N}, \tag{16}$$

where $\mathbf{W} \triangleq \mathbb{E}\{\mathbf{w}\mathbf{w}^H\}$ and $\mathbf{N} \triangleq \mathbb{E}\{\mathbf{n}\mathbf{n}^H\}$.

3.2. ZF Equalization

If the inverse of the operator matrix $\mathbf{B}$ is well-defined, then by designing the equalizer to have $\mathbf{Z} = \mathbf{B}^{-1}$, we will obtain

$$\mathbf{h} = \mathbf{Z}\mathbf{w} = \mathbf{f} + \mathbf{B}^{-1}\mathbf{n}. \tag{17}$$

However, when $\mathbf{B}$ is ill-conditioned, the ZF equalizer suffers from the well-known noise amplification problem.

3.3. MMSE Equalization

Given the operator $\mathcal{B}$ and the noise statistics and defining the error between the output and input signals as

$$e(\widehat{\boldsymbol{u}}) \triangleq h(\widehat{\boldsymbol{u}}) - f(\widehat{\boldsymbol{u}}),$$

the goal is to design an equalization operator $\mathcal{Z}$ such that it minimizes the mean square error, which is the trace of the following error covariance matrix

$$\mathbb{E}\{\mathbf{e}\mathbf{e}^H\} = \mathbf{Z}\mathbf{B}\mathbf{F}\mathbf{B}^H\mathbf{Z}^H + \mathbf{Z}\mathbf{N}\mathbf{Z}^H - \mathbf{Z}\mathbf{B}\mathbf{F} - \mathbf{F}\mathbf{B}^H\mathbf{Z}^H + \mathbf{F}. \tag{18}$$

From the orthogonality of the optimal error $e(\widehat{\boldsymbol{u}})$ to the observation $w(\widehat{\boldsymbol{u}})$ [11], the MMSE operator matrix is given by the familiar form

$$\mathbf{Z}^*_{\text{MMSE}} = \mathbf{F}\mathbf{B}^H(\mathbf{B}\mathbf{F}\mathbf{B}^H + \mathbf{N})^{-1}. \tag{19}$$

If the inverse of the operator matrix $\mathbf{B}$ is well-defined and in the absence of noise, $\mathbf{N} = \mathbf{0}$ and the equalizer reduces to the ZF form

$$\mathbf{Z}^*_{ZF} = \mathbf{B}^{-1}, \tag{20}$$

which from (15) results in perfect recovery $\mathbf{h} = \mathbf{f}$.

4. EQUALIZATION ON 2-SPHERE IN ACTION

4.1. Equalizing Spectral Truncation

Spectral truncation at degree L truncates the signal spherical harmonic coefficients to the first $L+1$ degrees. It takes the general spectral vector $\mathbf{f}$ and zeros out all spectral degrees $\ell > L$ to obtain

$$\mathbf{f}_{(L+1)^2} = \big((f)_0^0, (f)_1^{-1}, (f)_1^0, (f)_1^1, (f)_2^{-2}, \ldots, (f)_L^L, 0, 0, \ldots\big)'.$$

The corresponding operator matrix $\mathbf{B}$ has only non-zero diagonal elements as

$$b_{\ell,p}^{m,q} = \begin{cases} \delta_{\ell,p}^{m,q} & \ell \le L,\ |m| \le \ell, \\ 0 & \text{otherwise}, \end{cases} \tag{21}$$

in which case, the ZF equalizer is not well-defined. For MMSE equalizer using (19), we obtain

$$\mathbf{Z}^{*}_{\text{MMSE}} = \mathbf{F}\mathbf{B}^{H}(\mathbf{B}\mathbf{F}\mathbf{B}^{H} + \mathbf{N})^{-1} \tag{22}$$

$$= \mathbf{F}_{(L+1)^2}(\mathbf{F}_{(L+1)^2\times(L+1)^2} + \mathbf{N})^{-1}, \tag{23}$$

where $\mathbf{F}_{(L+1)^2}$ is the modified covariance matrix of the input signal $\mathbf{F}$ where only the first $(L+1)^2$ columns are kept unchanged and the remaining columns are replaced by zero. In addition, $\mathbf{F}_{(L+1)^2\times(L+1)^2}$ is the modified covariance matrix of the input signal $\mathbf{F}$ where only the first $(L+1)^2 \times (L+1)^2$ submatrix is kept unchanged and the remaining elements are replaced by zero. In the case where the input signal and noise are stochastically isotropic, $F^{m,q}_{\ell,p} = F_\ell \delta^{m,q}_{\ell,p}$ and also $N^{m,q}_{\ell,p} = N_\ell \delta^{m,q}_{\ell,p}$, then the equalizer operator matrix elements $z^{m,q}_{\ell,p}$ are explicitly given by

$$z^{m,q}_{\ell,p} = \begin{cases} \dfrac{F_\ell}{F_\ell + N_\ell}\delta^{m,q}_{\ell,p} & \ell \le L,\ |m| \le \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

This has a straightforward interpretation.

4.2. Equalizing Rotation

We denote the rotation operator as $\mathcal{D}(\varphi,\vartheta,\omega)$ parameterized in the Euler angles φ,ϑ,ω in the zyz convention, which corresponds to the rotation matrix $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ [9, Chapter 7]. The action of the rotation operator $\mathcal{D}(\varphi,\vartheta,\omega)$ can be realized through an inverse rotation of the coordinate system. That is,

$$\big(\mathcal{D}(\varphi,\vartheta,\omega)f\big)(\widehat{\boldsymbol{u}}) = f(\boldsymbol{R}^{-1}\widehat{\boldsymbol{u}}), \tag{25}$$

which means that the value of $\mathcal{D}(\varphi,\vartheta,\omega)f$ at point $\widehat{\boldsymbol{u}}$ is equal to the value of the original function f at point $\boldsymbol{R}^{-1}\widehat{\boldsymbol{u}}$ [9]. The rotation operator matrix is a block-diagonal matrix and is denoted by $\mathbf{D}$ with coefficients $d^{m,q}_{\ell,p}(\varphi,\vartheta,\omega)$ given by

$$d^{m,q}_{\ell,p}(\varphi,\vartheta,\omega) = D^{m,q}_{\ell}(\varphi,\vartheta,\omega)\delta_{\ell,p}, \tag{26}$$

where $D^{m,q}_{\ell}(\varphi,\vartheta,\omega)$ is the Wigner-D function [9]. The inverse rotation operator is $\mathcal{D}(-\omega,-\vartheta,-\varphi)$ and has the operator matrix equal to $\mathbf{D}^H$. That is, $\mathbf{D}^H\mathbf{D} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

In this case, the ZF equalizer is well defined which is the inverse rotation operator $\mathcal{D}(-\omega,-\vartheta,-\varphi)$. The MMSE equalizer is given by (19), but further simplification is not possible in the general case. However, if the input signal is isotropic and its covariance matrix $\mathbf{F}_{(L+1)^2}$ has a limited spectral degree L as

$$F^{m,q}_{\ell,p} = \begin{cases} \delta^{m,q}_{\ell,p} & \ell \le L,\ |m| \le \ell, \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

then (19) simplifies to

$$\mathbf{Z}^{*}_{\text{MMSE}} = \mathbf{F}_{(L+1)^2}\mathbf{D}^{H}(\mathbf{D}\mathbf{F}_{(L+1)^2}\mathbf{D}^{H} + \mathbf{N})^{-1} \tag{28}$$

$$= \mathbf{D}^{H}_{(L+1)^2}(\mathbf{I}_{(L+1)^2} + \mathbf{N})^{-1}, \tag{29}$$

where compared to $\mathbf{D}$ and $\mathbf{I}$ respectively, in $\mathbf{D}_{(L+1)^2}$ and $\mathbf{I}_{(L+1)^2}$ the first $(L+1)^2 \times (L+1)^2$ submatrix is kept unchanged and the remaining elements are replaced by zero. For isotropic noise, the MMSE equalizer matrix coefficients are given by

$$z^{m,q}_{\ell,p} = \begin{cases} \dfrac{\overline{D^{q,m}_{\ell}(\varphi,\vartheta,\omega)}\delta_{\ell,p}}{1+N_\ell} & \ell \le L,\ |m| \le \ell,\ |q| \le \ell \\[2ex] \dfrac{\overline{D^{q,m}_{\ell}(\varphi,\vartheta,\omega)}\delta_{\ell,p}}{N_\ell} & \text{otherwise.} \end{cases} \tag{30}$$

4.3. Equalizing Projection onto Azimuthally Symmetric Signals

Define the operator $\mathcal{B}$ to project a general signal f, which is not necessarily azimuthally symmetric, into the subspace of azimuthally symmetric signals $\mathcal{H}^0(\mathbb{S}^2) \subset L^2(\mathbb{S}^2)$, in which any $f(\theta,\phi) \in \mathcal{H}^0(\mathbb{S}^2)$ is only a function of co-latitude θ. Then we have

$$\big(\mathcal{B}f\big)(\widehat{\boldsymbol{u}}) = \sum_{\ell=0}^{\infty}(f)^0_\ell Y^0_\ell(\theta). \tag{31}$$

This projection operator matrix elements are therefore given by

$$b^{m,q}_{\ell,p} = \delta_{\ell,p}\delta_{q,0}\delta_{m,0}. \tag{32}$$

Therefore, the operator matrix is fully diagonal with non-zero diagonal elements only for order zero $m = q = 0$. The ZF equalizer is not well-defined. For MMSE equalizer, (19) specializes to

$$\mathbf{Z}^{*}_{\text{MMSE}} = \mathbf{F}\mathbf{B}^{H}(\mathbf{B}\mathbf{F}\mathbf{B}^{H} + \mathbf{N})^{-1} \tag{33}$$

$$= \mathbf{F}^0(\mathbf{F}^{0,0} + \mathbf{N})^{-1} \tag{34}$$

where $\mathbf{F}^0$ is the modified covariance matrix of the input signal $\mathbf{F}$ where only the elements in the zero-order columns, which are of the form $F^{m,0}_{\ell,p}$, are kept unchanged and all remaining columns are replaced by zero. That is, the elements of $\mathbf{F}^0$ are of the form $F^{m,0}_{\ell,p}\delta_{q,0}$. Similarly, $\mathbf{F}^{0,0}$ is a further modified version of $\mathbf{F}^0$ where only zero-order rows of $\mathbf{F}^0$ are kept unchanged and all remaining rows are replaced by zero. That is, the elements of $\mathbf{F}^{0,0}$ are of the form $F^{0,0}_{\ell,p}\delta_{q,0}\delta_{m,0}$.

In the case where input signal and noise are stochastically isotropic, the equalizer operator matrix elements $z^{m,q}_{\ell,p}$ are explicitly given by

$$z^{m,q}_{\ell,p} = \frac{F_\ell}{F_\ell + N_\ell}\delta_{\ell,p}\delta_{q,0}\delta_{m,0}. \tag{35}$$

4.4. Equalizing Spectral Weighting

Spectral weighting of the signal is performed through an operator with kernel given by [8]

$$B(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}}) \triangleq \sum_{\ell,m} s^m_\ell Y^m_\ell(\widehat{\boldsymbol{u}})\overline{Y^m_\ell(\widehat{\boldsymbol{v}})}, \tag{36}$$

and a diagonal operator matrix $\mathbf{B}$ with elements of the form

$$b^{m,q}_{\ell,p} = s^m_\ell \delta^{m,q}_{\ell,p}, \tag{37}$$

resulting in the spectrally weighted output signal $g = \mathcal{B}f$ with spherical harmonic Fourier coefficients given by

$$(g)^m_\ell = s^m_\ell\,(f)^m_\ell. \tag{38}$$

ZF equalizer for spectral weighting operator is simply given by

$$z^{m,q}_{\ell,p} = \frac{1}{s^m_\ell}\delta^{m,q}_{\ell,p}, \quad s^m_\ell \neq 0. \tag{39}$$

For MMSE equalizer, (19) specializes to

$$\mathbf{Z}^{*}_{\text{MMSE}} = \mathbf{F}\mathbf{B}^{H}(\mathbf{B}\mathbf{F}\mathbf{B}^{H} + \mathbf{N})^{-1} \tag{40}$$

$$= \mathbf{\Omega}(\mathbf{\Lambda} + \mathbf{N})^{-1}, \tag{41}$$

where the elements of matrices $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ are, respectively, given by

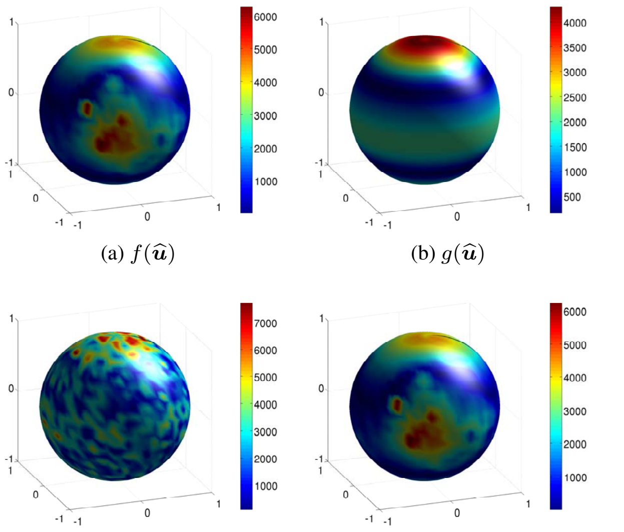
$$\omega^{m,q}_{\ell,p} = F^{m,q}_{\ell,p}\overline{s^m_\ell}, \tag{42}$$

$$\lambda^{m,q}_{\ell,p} = s^m_\ell F^{m,q}_{\ell,p}\overline{s^q_p}. \tag{43}$$

4.5. Equalizing Isotropic Spectral Weighting

The spectral weighting with isotropic kernel

$$B(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}}) \triangleq \sum_{\ell=0}^{\infty} s_\ell \frac{2\ell+1}{4\pi}P^0_\ell(\widehat{\boldsymbol{u}}.\widehat{\boldsymbol{v}}), \tag{44}$$



(a) $f(\widehat{\boldsymbol{u}})$ (b) $g(\widehat{\boldsymbol{u}})$ (c) $w(\widehat{\boldsymbol{u}})$ (d) $h(\widehat{\boldsymbol{u}})$

Fig. 2: (a) The signal $f(\widehat{\boldsymbol{u}})$ as Mars topographic map, band-limited at $L = 32$. (b) The projected azimuthally symmetric signal $g(\widehat{\boldsymbol{u}}) = \big(\mathcal{B}f\big)(\widehat{\boldsymbol{u}})$. (c) The noise corrupted signal $w(\widehat{\boldsymbol{u}}) = g(\widehat{\boldsymbol{u}}) + n(\widehat{\boldsymbol{u}})$ equalized using MMSE equalizer in (33) to obtain (d) the equalized signal $h(\widehat{\boldsymbol{u}})$.

has been referred to as spectral smoothing in [1] and is equivalent to the isotropic convolution using the azimuthally symmetric function $k(\widehat{\boldsymbol{u}}) \in \mathcal{H}^0(\mathbb{S}^2)$ given by [5,8]

$$k(\widehat{\boldsymbol{u}}) = \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}}\, s_\ell Y_\ell(\widehat{\boldsymbol{u}}). \tag{45}$$

With an isotropic condition for the kernel (when $s_\ell^m = s_\ell$), the ZF and MMSE solutions given in (39) and (40) respectively also hold.

Futhermore, when the input signal and noise are stochastically isotropic, the equalizer operator matrix elements $z_{\ell,p}^{m,q}$ are

$$z_{\ell,p}^{m,q} = \frac{F_\ell \overline{s_\ell}}{F_\ell |s_\ell|^2 + N_\ell}\, \delta_{\ell,p}^{m,q} \tag{46}$$

which is identical to the one reported in [6].

5. ILLUSTRATION

In order to illustrate the operation of proposed equalization framework, we equalize the projection of the signal $f(\widehat{\boldsymbol{u}}) \in L^2(\mathbb{S}^2)$ into the subspace of azimuthally symmetric signals. In order to quantify the quality of any signal $h(\widehat{\boldsymbol{u}})$ with respect to the original (reference) signal $f(\widehat{\boldsymbol{u}})$, define the signal-to-noise ratio (SNR) for the signal $h(\widehat{\boldsymbol{u}})$ as $\text{SNR}^h = 20\log\frac{\|f\|}{\|f-h\|}$. We consider the Mars topographic map (height above geoid) as a signal $f(\widehat{\boldsymbol{u}})$ band-limited at $L = 32$ shown in Fig. 2(a). The projected azimuthally symmetric signal $g(\widehat{\boldsymbol{u}}) = \big(\mathcal{B}f\big)(\widehat{\boldsymbol{u}})$ is shown in Fig. 2(b), where $g(\widehat{\boldsymbol{u}}) \in \mathcal{H}^0(\mathbb{S}^2)$ and the operator matrix elements are given in (32). Due to projection, the SNR of the signal $g(\widehat{\boldsymbol{u}})$ is $\text{SNR}^g = 6.54$dB. The noise corrupted signal $w(\widehat{\boldsymbol{u}}) = g(\widehat{\boldsymbol{u}}) + n(\widehat{\boldsymbol{u}})$ is shown in Fig. 2(c) with $\text{SNR}^g = -1.10$dB. The noise is added in spectral domain and is considered to be Gaussian and anisotropic but uncorrelated, that is, $N_{\ell,p}^{m,q} = \mathbb{E}\big\{(n)_\ell^m \overline{(n)_p^q}\big\} = \big(\sigma_\ell^m\big)^2 \delta_{\ell,p}^{m,q}$, where $\sigma_\ell^m > 0$ denotes the standard deviation of $(n)_\ell^m$ and is randomly chosen with uniform distribution $[0, K]$, where $K > 0$ is a constant which determines the strength (energy) of the noise. The equalized (recovered) signal $h(\widehat{\boldsymbol{u}})$ is obtained using the MMSE equalizer given in (33). Since the signal $f(\widehat{\boldsymbol{u}})$ is deterministic, the covariance matrix is obtained as $\mathbf{F} = \mathbb{E}\big\{\mathbf{f}\mathbf{f}^H\big\} = \mathbf{f}\mathbf{f}^H$. The recovered signal $h(\widehat{\boldsymbol{u}})$ is shown in Fig. 2(d) with much improved signal-to-noise ratio, $\text{SNR}^h = 43.14$dB, which is due to the fact that the equalizer exploits the non-isotropic information left in $\mathbf{F}^0$ and $\mathbf{F}^{0,0}$ in (34) to determine the recovered signal. We note that the illustration here is naive and presented merely to demonstrate the proposed equalization framework. Application to real data and comparison with other non-linear signal estimation techniques serve as future research directions.

6. CONCLUSIONS

In this paper, we have developed the general ZF and MMSE linear equalizers for the recovery of anisotropic signals (or stochastic processes) on the 2-sphere that are subject to linear distortions and noise. We have considered the distortions as bounded operators and formulated the proposed equalizers for operations including spectral truncation, rotation, projection and spectral weighting. We have also illustrated the operation of proposed equalization framework through an example. The comparison of the proposed framework with the existing non-linear signal recovery techniques presents an open problem for further work.

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