SPHERICAL HARMONIC TRANSFORM FOR MINIMUM DIMENSIONALITY REGULAR GRID SAMPLING ON THE SPHERE

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ABSTRACT

We develop a method to compute spherical harmonic transform (SHT) of a band-limited signal on the sphere discretized over a minimum dimensionality regular sampling grid on the sphere. For the computation of SHT of a signal band-limited at L, the proposed method requires L^2 number of samples on a regular grid composed of L iso-latitude rings of samples with only L samples in each ring along longitude. Since a signal band-limited at L is represented by L^2 degrees of freedom in the spectral (spherical harmonic) domain, the proposed method requires the minimal number of samples for the computation of SHT. In comparison to the other schemes that require $2L-1$ samples along each iso-latitude ring, we show that the SHT can be computed, by exploiting the structure of spectral domain, from only L samples in each iso-latitude ring. We also analyse the numerical accuracy and the computational complexity of our proposed SHT for a regular grid with equiangular sampling. We demonstrate, through numerical experiments, that the proposed SHT is sufficiently accurate for band-limits of interest in diffusion magnetic resonance imaging.

Index Terms— spherical harmonic transform; sampling; band-limited signals; unit sphere.

1. INTRODUCTION

We consider a Hilbert space $L^2(\mathbb{S}^2)$ formed by a set of square integrable complex valued functions defined on the two dimensional unit sphere (or 2-sphere), denoted by $\mathbb{S}^2$. The space $L^2(\mathbb{S}^2)$ is equipped with the inner product defined for two functions f and h defined on $\mathbb{S}^2$ as [1]

$$\langle f,h\rangle \triangleq \int_{\mathbb{S}^2} f(\theta,\phi)\overline{h(\theta,\phi)}\,\sin\theta\, d\theta\, d\phi, \tag{1}$$

where $\theta \in [0,\pi]$ denotes the latitude that is measured from positive z-axis, $\phi \in [0,2\pi)$ denotes the longitude that is measured from the positive x-axis in the $x-y$ plane, $\overline{(\cdot)}$ denotes the complex conjugate operation, $\sin\theta\, d\theta\, d\phi$ denotes the differential area element on the sphere and the integration is carried out over $\mathbb{S}^2$. The inner product in (1) induces a norm $\|f\| \triangleq \langle f,f\rangle^{1/2}$, and the functions with finite induced norm are referred as signals on the sphere.

The development of signal processing techniques for signals defined on $\mathbb{S}^2$ finds applications in various fields of science and engineering (e.g., [1–7])). In these applications, the signal is analysed either in the spatial domain or spectral domain or both. The spectral domain is enabled by the spherical harmonic transform (SHT) – the well-known counterpart of the Fourier transform for signals on the sphere [1]. It is desirable that the SHT of a signal can be computed from the least number of measurements (samples) of the signal taken over the sphere. This is particularly important in applications, for example, diffusion magnetic resonance imaging (dMRI) [8, 9], where the time required to acquire a single measurement is large.

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1.1. Relation to Prior Work

Many sampling schemes on the sphere, supported by accurate computation of SHT, have been proposed in the literature. In this work, we restrict our attention to the sampling schemes [10–13] which permit the accurate computation of the SHT of a signal that is band-limited at L (formally defined in Section 2) and are composed of samples on the sphere taken over a regular (or equiangular) grid - a grid formed by samples along latitude and longitude. We note that the minimum number of samples, denoted by $N_{\rm O}$, attainable by any sampling scheme that allows the accurate computation of SHT of a band-limited signal is given by $N_{\rm O} = L^2$, which is revealed by the degrees of freedom required to represent a band-limited signal in the spectral domain [13, 14].

An exact method to compute the SHT was first developed in 1994 in [10] for an equiangular sampling scheme comprised of $2L$ iso-latitude rings of samples with $2L-1$ samples in each ring along longitude. Thus the total number of samples required by the sampling scheme proposed in [10] is of the order of $4L^2$. Recently, in 2011 in [13], a new sampling theorem has been proposed for an equiangular sampling which asymptotically requires $2L^2$ number of samples. Moreover, the Gauss-Legendre quadrature on the sphere [15, 16] may also be used to construct a sampling theorem and exact SHT from asymptotically $2L^2$ samples on the sphere. Although the placement of samples dictated by the Gauss-Legendre quadrature is on a regular grid, the samples along latitude are not equiangular. The SHT using the least squares approach have also been developed for equiangular sampling schemes [17, 18] that also require $2L^2$ samples for the accurate computation of SHT. A least squares method can also be used to compute SHT for the L^2 samples, but it may not yield accurate SHT as we show later in the paper. More recently, a sampling scheme, referred as optimal spatial dimensionality sampling scheme, has been proposed in [14] that requires only L^2 (minimum) number of samples on the sphere and allows accurate computation of SHT. However, the samples in the scheme are not defined on a regular grid.

1.2. Contributions

In all of the developments for sampling on the regular grid on the sphere, $2L-1$ number of samples are taken along longitude for each sample along latitude with an objective to avoid aliasing in the computation of SHT, where the aliasing occurs due to the method chosen to compute SHT and should *not* be confused with the aliasing due to undersampling of the signal. In this work, we show that the aliasing errors can be avoided with much fewer samples by exploiting the structure of the spectral domain. We develop a method to compute SHT of the signal band-limited at L and sampled on a regular sampling (or equiangular sampling) grid that is composed of L iso-latitude rings of samples with only L samples in each ring along longitude. Thus, the SHT transform can be computed from the minimum, L^2, number of samples taken over a regular grid on the sphere. We develop a matrix formulation for the proposed SHT and show that the aliasing errors can be avoided by solving a series of linear systems. We also evaluate the numerical accuracy and the computational complexity of our proposed SHT and show that the SHT is sufficiently accurate for band-limits of interest in diffusion magnetic resonance imaging (dMRI) application [8, 9]. In comparison to the proposed SHT, we note that SHT for an optimal dimensionality sampling scheme proposed in [14] is numerically superior but requires samples on a non-regular grid.

The rest of the paper is organized as follows. We review the harmonic analysis in Section 2. In Section 3, we develop SHT for minimum dimensionality sampling scheme on the sphere. The numerical accuracy and computational complexity of the proposed SHT is analysed in Section 4. Finally, conclusions are made in Section 5.

2. HARMONIC ANALYSIS ON THE SPHERE

Spherical harmonics, denoted by $Y_\ell^m(\theta,\phi)$, are defined for integer degree $\ell \geq 0$ and integer order $m \in [-\ell, \ell]$ as [1]

$$Y_\ell^m(\theta,\phi) \triangleq \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi},$$

where P_ℓ^m denotes the associated Legendre function [1]. Spherical harmonics form archetype complete orthonormal set of basis functions for $L^2(\mathbb{S}^2)$, and therefore we can expand any signal $f \in L^2(\mathbb{S}^2)$ as

$$f(\hat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(f)_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}), \tag{2}$$

where $(f)_\ell^m$ denotes the spherical harmonic coefficient of degree ℓ and order m, and is given by the spherical harmonic transform (SHT):

$$(f)_\ell^m \triangleq \langle f, Y_\ell^m\rangle, = \int_{\mathbb{S}^2} f(\theta,\phi)\overline{Y_\ell^m(\theta,\phi)}\,\sin\theta\, d\theta\, d\phi. \tag{3}$$

The spherical harmonic coefficients $(f)_\ell^m$ form the spectral domain representation of the signal. The reconstruction of signal on the sphere from its spectral domain representation (spherical harmonic coefficients), given in (2), is referred to as inverse SHT. The signal f is said to be *band-limited at degree* L if $(f)_\ell^m = 0$, $\forall \ell > L$ and the set of all such band-limited signals form an L^2 dimensional subspace of $L^2(\mathbb{S}^2)$, denoted by $\mathcal{H}_L$. Thus, any band-limited signal $f \in \mathcal{H}_L$ has only L^2 degrees of freedom.

3. SPHERICAL HARMONIC TRANSFORM FOR REGULAR GRID SAMPLING

3.1. Sampling Scheme Structure

We propose an iso-latitude sampling of the sphere, denoted by $\mathfrak{S}_L^K$, composed L iso-latitude rings of samples with only K samples in each ring along longitude. Let θ_t for $t = 0, 1, \ldots, L-1$ denotes the position of samples along latitude. We develop SHT for the arbitrary placement of samples along latitude. We analyse the proposed SHT for equiangular placement of samples along latitude later in the paper. For the discretization along longitude for each iso-latitude ring, we choose K equiangular samples with sample locations given by

$$\phi_p = \frac{2\pi(t+1)}{K}, \quad p = 0, 1, \ldots, K-1. \tag{4}$$

We show later in the paper that the equiangular placement of samples along longitude is an optimal choice. When $K = L$, the total number of samples in the proposed sampling $\mathfrak{S}_L^L$ scheme is $N_{\rm O} = L^2$ and the sampling scheme is referred to as a minimal dimensionality sampling scheme on the sphere.

3.2. Spherical Harmonic Transform

We develop a method to compute SHT of a band-limited signal $f \in \mathcal{H}_L$ discretized over the sampling scheme $\mathfrak{S}_L^K$. Changing the order of summation in (2), we can write a sampled signal $f(\theta_t,\phi_p)$ as

$$f(\theta_t,\phi_p) = \sum_{m=-(L-1)}^{L-1} e^{im\phi_p}G_m(\theta_t), \tag{5}$$

with

$$G_m(\theta_t) \triangleq \sum_{\ell=|m|}^{L-1}(f)_\ell^m \tilde{P}_\ell^m(\cos\theta_t), \tag{6}$$

where $\tilde{P}_\ell^m(\theta_t) \triangleq Y_\ell^m(\theta_t, 0) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta_t)$ denotes scaled associated Legendre functions. Note that $G_m(\theta_t)$ depends on the m-th order spherical harmonic coefficients. For the sampled signal $f(\theta_t,\phi_p)$ and order $|m| < L$, define $A_m(\theta_t)$ as K point discrete Fourier transform (DFT) along ϕ, given by

$$A_m(\theta_t) \triangleq \sum_{p=0}^{K-1} f(\theta_t,\phi_t)e^{-im\phi_p}. \tag{7}$$

Since $|m| < L$, $A_m(\theta_t) = KG_m(\theta_t)$ for $K \geq 2L-1$ (Nyquist criterion), which can be verified by substituting $f(\theta_t,\phi_p)$, given in (5), in (7), and employing the orthogonality of discrete complex exponentials. Thus, $G_m(\theta_t)$ can be computed *exactly* from the sampled signal over the grid $\mathfrak{S}_L^K$ when at least $K = 2L-1$ samples are taken along ϕ in each iso-latitude ring. Once $G_m(\theta_t)$ is computed for each θ_t, the spherical harmonic coefficients can be computed using the Gauss-Legendre quadrature [15, 16] or the quadrature rules dictated by sampling theorems [10, 13]. If the samples are not equiangular along longitude, the orthogonality of discrete complex exponentials cannot be directly applied. Nevertheless, $G_m(\theta_t)$ can still be computed from $K \geq 2L-1$ samples along longitude as the Nyquist criterion is satisfied. We propose that an equiangular placement along longitude is an optimal choice as this allows to use the orthogonality of complex exponentials.

When $K < 2L-1$, $G_m(\theta_t)$ cannot be computed from the sampled signal due to the aliasing along longitude. We again highlight that the aliasing occurs due to the formulation of SHT given in (5) and (7). We show that this aliasing can be avoided by exploiting the structure of the spectral domain and the spherical harmonic coefficients can be computed from the samples of the signal taken over $\mathfrak{S}_L^L$. For $K = L$, the substitution of the sampled signal $f(\theta_t,\phi_p)$, given in (5), in (7) yields

$$A_m(\theta_t) = LG_m(\theta_t) + \delta_{m,0}\, LG_{m-L}(\theta_t), \quad m \geq 0, \tag{8}$$

where $\delta_{m,0}$ is the Kronecker delta function: $\delta_{m,0} = 1$ for $m = 0$ and is zero otherwise. The formulation in (8) indicates that $A_m(\theta_t)$

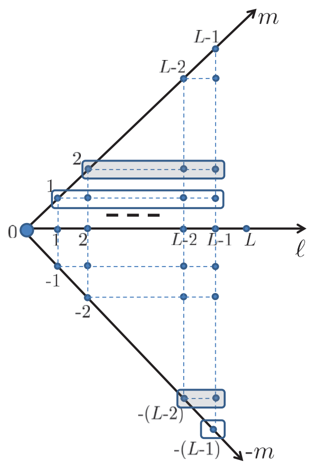


Fig. 1: The graphical representation of the spectral domain (formed by spherical harmonic coefficients) of a signal band-limited at L. For each $|m| < L$, $L - m$ number of spherical harmonic coefficients of order m and m number of coefficients of order $m - L$ contribute in $A_m(\theta_t)$. This is depicted for $m = 1$ and $m = 2$.

contains the contribution of spherical harmonic coefficients of orders m and $m - L$. As an example, this is depicted in Fig. 1 that shows the graphical representation of the spectral domain of a signal band-limited at L and the components contribute in $A_m(\theta_t)$ for $m = 1$ and $m = 2$. For each $m \geq 0$, there are $L - m$ number of coefficients of order m and m number of coefficients of order $m - L$, therefore the total number of coefficients to be recovered from $A_m(\theta_t)$ for each m is always L. Since we have $A_m(\theta_t)$ for each θ_t, $t =\in [0, L-1]$, L number of spherical harmonic coefficients can be recovered from $A_m(\theta_t)$ by setting up a linear system.

Define a vector $\mathbf{f}_m$ for $m \geq 0$ containing spherical harmonic coefficients of orders m and $m - L$ given by

$$\mathbf{f}_m = \big[(f)^m_m,\, (f)^m_{m+1},\, \ldots,\, (f)^m_{L-1},\, (f)^{m-L}_{|m-L|},\, (f)^{m-L}_{|m-L|+1},\, \ldots,\, (f)^{m-L}_{L-1}\big]^T, \tag{9}$$

and a vector $\mathbf{a}_m$ containing $A_m(\theta_t)$ for each θ_t given by

$$\mathbf{a}_m = \frac{1}{L}\big[A_m(\theta_0),\, A_m(\theta_1),\, \ldots,\, A_m(\theta_{L-1})\big]^T. \tag{10}$$

By defining a matrix $\mathbf{P}_m$, for $m \geq 0$, of size $L \times L$ with t-th row given by

$$[\mathbf{P}_m]_{t,:} \triangleq \big[\tilde{P}^m_m(\theta_t),\, \tilde{P}^m_{m+1}(\theta_t),\, \ldots,\, \tilde{P}^m_{L-1}(\theta_t),\, \tilde{P}^{m-L}_{|m-L|}(\theta_t),\, \tilde{P}^{m-L}_{|m-L|+1}(\theta_t),\, \ldots,\, \tilde{P}^{m-L}_{L-1}(\theta_t)\big], \tag{11}$$

we can write (8), noting (6), as

$$\mathbf{P}_m \mathbf{f}_m = \mathbf{a}_m, \quad m \geq 0. \tag{12}$$

The solution of a system in (12) for each m yields the spherical harmonic coefficients of order m and order $m - L$. Thus, we can recover all of the spherical harmonic coefficients by solving a series of systems of the form, given in (12), for all non-negative orders $0 \leq m < L$. We summarise the computation of proposed SHT of a band-limited signal $f \in \mathcal{H}_L$ sampled over minimal dimensionality sampling scheme $\mathfrak{S}^L_L$ as a two step procedure:

1. Compute $A_m(\theta_t)$, given in (7), for all $0 \leq m < L$ by taking L point DFT along longitude for each ring placed at each θ_t.
2. Solve (12) for each $0 \leq m < L$ to obtain the spherical harmonic coefficients (contained in a vector $\mathbf{f}_m$) of order m and order $m - L$.

We solve (12) using least squares method in this work, which requires the matrix $\mathbf{P}_m$ to be well-conditioned. We analyse the numerical accuracy and computational complexity of the spherical harmonic transform later in the paper.

Remark 1. *This is due to the structure of spectral domain formed by spherical harmonic coefficients that there are always L number of coefficients contribute to $A_m(\theta_t)$ in* (8)*, which allows us to recover these coefficients from a series of systems of the form, given in* (12).

Remark 2. *We also highlight that the proposed method to compute SHT can only be used for odd band-limit L. When the band-limit L is even, the columns of the matrix $\mathbf{P}_m$ are composed of scaled associated Legendre functions $\tilde{P}^m_\ell(\theta_t)$ of orders $m = \frac{L}{2}$ and $m - L = -\frac{L}{2}$ and degrees $\frac{L}{2} \leq \ell < L-1$ evaluated over the sample points θ_t, $t = 0, 1, \ldots, L-1$. Since $\tilde{P}^{-m}_\ell(\theta_t) = (-1)^m \tilde{P}^m_\ell(\theta_t)$, the matrix $\mathbf{P}_{\frac{L}{2}}$ becomes singular for even band-limit L, and therefore the spherical harmonic transform cannot be computed using the proposed method.*

3.3. Inverse Spherical Harmonic Transform

The inverse SHT yields the signal on the sphere over the proposed minimal dimensionality sampling scheme $\mathfrak{S}^L_L$ from its spherical harmonic coefficients. For the proposed sampling scheme, the inverse SHT can be obtained using the formulation of the sampled signal given in (5) by first computing $G_m(\theta_t)$ in (6) for each $|m| < L$.

3.4. Computation of Scaled Associated Legendre Functions

Both SHT and inverse SHT require the computation of scaled associated Legendre function $\tilde{P}_m(\theta_t)$ for each θ_t and for all degrees $\ell < L$ and orders $|m| < L$. Since $\tilde{P}^m_\ell(\theta_t) = (-1)^m \tilde{P}^{-m}_\ell(\theta_t)$, we only compute $\tilde{P}_m(\theta_t)$ for non-negative orders $0 \leq m < L$. Since the entries in the matrix $\mathbf{m}$ requires to compute $\tilde{P}^m_\ell(\theta_t)$ for $|m| \leq \ell < L$ and $\tilde{P}^{m-L}_\ell(\theta_t)$ for $|m - L| \leq \ell < L$, we adopt the three-term recursion [14, 19], for the computation of scaled associated Legendre functions, which grows with degree ℓ and computes $\tilde{P}^m_\ell(\theta_t)$ for all $|m| \leq \ell < L - 1$ for a given m. Furthermore, the recurrence relation that grows with degree ℓ for a given m is also a suitable choice for inverse SHT, where the computation of $G_m(\theta_t)$ in (6) requires $\tilde{P}^m_\ell(\theta_t)$ to be computed for each θ_t and for all degrees $|m| \leq \ell < L$ for a given order m.

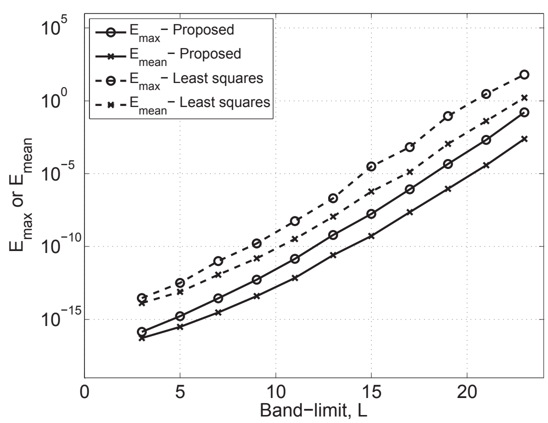
4. ANALYSIS OF PROPOSED SHT

4.1. Accuracy Analysis

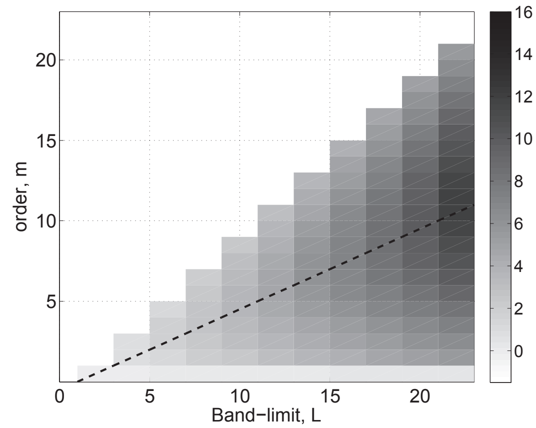
We analyse the accuracy of the proposed SHT for minimal dimensionality regular grid sampling scheme $\mathfrak{S}^L_L$, where we choose equiangular placement of iso-latitude rings for simplicity, that is,

$$\theta_t = \frac{\pi(t+1)}{L+1}, \quad t = 0, 1, \ldots, L-1. \tag{13}$$

A SHT for any sampling scheme is accurate if the SHT (or inverse SHT) of a band-limited signal followed by the inverse SHT (or SHT) yields the same band-limited signal. In order to evaluate the numerical accuracy of the proposed SHT, we generate a complex valued test signal f_{T} over the proposed sampling grid $\mathfrak{S}^L_L$, where the real and complex components of the value at each sample are chosen from a uniform distribution on the interval $[-1, 1]$. We apply the SHT followed by the inverse SHT on the signal, to obtain the reconstructed signal f_{R}. We repeat this experiment 20 times and obtain the average values for the maximum and mean reconstruction errors, given



(a) Errors, E_{max} or E_{mean}



(b) Condition number, $\log_{10} \kappa_m$

Fig. 2: (a) The reconstruction errors, E_{max} and E_{mean}, for the proposed SHT and the least squares based computation for odd band-limits in the range $2 \leq L \leq 23$. (b) The condition number as $\log_{10} \kappa_m$, of the matrix $\mathbf{P}_m$ for $0 \leq m < L$ and odd band-limits in the range $2 \leq L \leq 23$. The condition number κ_m is maximum for $m = \frac{L+1}{2}$ and $m = \frac{L-1}{2}$ for the band-limit L as indicated by a dashed line.

by

$$E_{\text{max}} \triangleq \max |f_{\text{T}}(\theta_t, \phi_p) - f_{\text{R}}(\theta_t, \phi_p)|, \quad 0 \leq t, p < L \tag{14}$$

$$E_{\text{mean}} \triangleq \frac{1}{L^2} \sum_{t=0}^{L-1} \sum_{p=0}^{L-1} |f_{\text{T}}(\theta_t, \phi_p) - f_{\text{R}}(\theta_t, \phi_p)|. \tag{15}$$

Since a least squares method can also be employed to compute SHT by setting up a linear system that determines L^2 number of spherical harmonic coefficients from L^2 samples, we also use least squares approach to compute SHT in our experiments and record E_{max} and E_{mean}. The reconstruction errors for the proposed SHT and the least squares based computation of SHT are plotted in Fig. 2(a) for odd band-limits in the range $2 \leq L \leq 23$, where it is evident that the proposed method to compute SHT method is more accurate than the least squares based SHT. It can also be observed that both the maximum error E_{max} and the mean error E_{mean}, for the proposed SHT, grows with the band-limit L, which is due to the reason that the matrices $\mathbf{P}_m$ for $0 \leq m < L$ does not remain well-conditioned for large odd band-limit L. The condition number (ratio of the largest eigenvalue to the smallest eigenvalue), denoted by κ_m, of the matrix $\mathbf{P}_m$ for $0 \leq m < L$ and odd band-limits in the range $2 \leq L \leq 23$ is plotted in Fig. 2(b), which illustrates that the condition number κ_m increases with the band-limit L for a given m. It can also be noted that the condition number κ_m is maximum for $m = \frac{L+1}{2}$ and $m = \frac{L-1}{2}$ for the band-limit L. The matrix $\mathbf{P}_m$, composed of scaled associated Legendre functions of order m and order $m-L$, becomes ill-conditioned as the associated Legendre functions of different orders are not orthogonal.

Since the SHT becomes inaccurate for large band-limits, the proposed sampling scheme is not suitable for geophysical or cosmological applications [20], where the band-limit L of the signal is of the order 10^2 – 10^4. Since the maximum error E_{max} is on the order of 10^{-10} and 10^{-5} for the band-limits $L = 11$ and $L = 21$ respectively, the proposed SHT is useful in the dMRI application [9], where the band-limit is of the order 10–20. We also highlight that the matrix $\mathbf{P}_m$ also depends on the samples along latitude, which are placed with equiangular spacing in this work. The consideration of alternative choices for the placement of samples along latitude and the optimisation of placement of samples, such that condition number κ_m of the matrix $\mathbf{P}_m$ for each $0 \leq m < L$ is improved, remains a subject of future work.

4.2. Computational Complexity Analysis

We discuss the computational complexity of SHT and inverse SHT. For the computation of proposed SHT, the fast Fourier transform (FFT) can be used to compute $A_m(\theta_t)$ for all $0 \leq m < L$ and all θ_t, with computational complexity $O(L^2 \log_2 L)$. Next, the entries of the matrix $\mathbf{P}_m$ of size $L \times L$ are computed recursively in $O(L^2)$ time for each m. Finally, we determine the spherical harmonic coefficients contained in a vector $\mathbf{f}_m$ by solving the linear system in (12) with complexity $O(L^3)$ for each m. Since this step needs to be repeated for each $0 \leq m < L$, the overall asymptotic complexity of the proposed SHT scales as $O(L^4)$, which is similar to the complexity of the spherical harmonic transform for the sampling scheme that requires minimum number of samples on iso-latitude (irregular) grid [14]. The inverse SHT is formulated in (5) by exploiting the iso-latitude structure of the sampling scheme and using the separation of variables, that allows us to compute the summation of $O(L)$ over the L^2 samples with complexity $O(L^3)$, which is similar to the complexity of the transforms that exist in literature for different sampling schemes on the sphere.

5. CONCLUSIONS

In this paper, a method has been proposed to compute spherical harmonic transform (SHT) of a signal band-limited at L from its L^2 number of samples taken over a regular grid composed of L iso-latitude rings of samples with only L samples in each ring along longitude. The proposed method requires minimum number of samples for the accurate computation of SHT as the signal band-limited at L is represented by L^2 degrees of freedom in the spectral domain. In comparison to the existing regular grid sampling schemes on the sphere that require $2L-1$ samples along longitude in each iso-latitude ring to avoid aliasing errors, we have considered L samples in each iso-latitude ring and shown that the aliasing errors can be avoided by exploiting the structure of the spectral domain. We have also analysed the numerical accuracy and the computational complexity of the proposed SHT, where we have shown that the proposed method to compute SHT is more accurate than the least squares based SHT.

6. REFERENCES

[1] R. A. Kennedy and P. Sadeghi, *Hilbert Space Methods in Signal Processing*. Cambridge, UK: Cambridge University Press, Mar. 2013.

[2] J. Antoine and P. Vandergheynst, "Wavelets on the n-sphere and related manifolds," *J. Math. Phys.*, vol. 39, p. 3987, 1998.

[3] P. Sadeghi, R. A. Kennedy, and Z. Khalid, "Commutative anisotropic convolution on the 2-sphere," *IEEE Trans. Signal Process.*, vol. 60, no. 12, pp. 6697–6703, Dec. 2012.

[4] Z. Khalid, S. Durrani, R. A. Kennedy, and P. Sadeghi, "On the construction of low-pass filters on the unit sphere," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process., ICASSP'2011*, Prague, Czech Republic, May 2011, pp. 4356–4359.

[5] Y. Wiaux, J. D. McEwen, P. Vandergheynst, and O. Blanc, "Exact reconstruction with directional wavelets on the sphere," *Mon. Not. R. Astron. Soc.*, vol. 388, no. 2, pp. 770–788, 2008.

[6] M. A. Wieczorek and F. J. Simons, "Localized spectral analysis on the sphere," *Geophys. J. Int.*, vol. 162, no. 3, pp. 655–675, Sep. 2005.

[7] P. Audet, "Toward mapping the effective elastic thickness of planetary lithospheres from a spherical wavelet analysis of gravity and topography," *Physics of the Earth and Planetary Interiors*, vol. 226, pp. 48–82, Jan. 2014.

[8] E. Caruyer, C. Lenglet, G. Sapiro, and R. Deriche, "Design of multishell sampling schemes with uniform coverage in diffusion MRI," *Magn. Reson. Med.*, vol. 69, no. 6, pp. 1534–1540, Jun. 2013.

[9] C. P. Hess, P. Mukherjee, E. T. Han, D. Xu, and D. B. Vigneron, "Q-ball reconstruction of multimodal fiber orientations using the spherical harmonic basis," *Magn. Reson. Med.*, vol. 56, no. 1, pp. 104–117, Jul. 2006.

[10] J. R. Driscoll and D. M. Healy, Jr., "Computing Fourier transforms and convolutions on the 2-sphere," *Adv. Appl. Math.*, vol. 15, no. 2, pp. 202–250, Jun. 1994.

[11] K. M. Górski, E. Hivon, A. J. Banday, B. D. Wandelt, F. K. Hansen, M. Reinecke, and M. Bartelmann, "HEALPix: A Framework for High-Resolution Discretization and Fast Analysis of Data Distributed on the Sphere," *Astrophys. J.*, vol. 622, pp. 759–771, Apr. 2005.

[12] R. G. Crittenden and N. G. Turok, "Exactly azimuthal pixelizations of the sky," *Arxiv preprint astro-ph/9806374*, 1998.

[13] J. D. McEwen and Y. Wiaux, "A novel sampling theorem on the sphere," *IEEE Trans. Signal Process.*, vol. 59, no. 12, pp. 5876–5887, Dec. 2011.

[14] Z. Khalid, R. A. Kennedy, and J. D. McEwen, "An optimal-dimensionality sampling scheme on the sphere with fast spherical harmonic transforms," *IEEE Trans. Signal Process.*, vol. 62, no. 17, pp. 4597–4610, Sep. 2014.

[15] W. Skukowsky, "A quadrature formula over the sphere with application to high resolution spherical harmonic analysis," *J. Geodesy*, vol. 60, no. 1, pp. 1–14, Mar. 1986.

[16] A. G. Doroshkevich, P. D. Naselsky, O. V. Verkhodanov, D. I. Novikov, V. I. Turchaninov, I. D. Novikov, P. R. Christensen, and Chiang, "Gauss Legendre Sky Pixelization (GLESP) for CMB maps," *Int. J. Mod. Phys. D.*, vol. 14, no. 02, pp. 275–290, Feb. 2005.

[17] N. Sneeuw, "Global spherical harmonic analysis by least-squares and numerical quadrature methods in historical perspective," *Geophys. J. Int.*, vol. 118, no. 3, pp. 707–716, Sep. 1994.

[18] J. Blais and M. Soofi, "Spherical harmonic transforms using quadratures and least squares," in *Computational Science ICCS 2006*, ser. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2006, vol. 3993, pp. 48–55.

[19] P. J. Kostelec and D. N. Rockmore, "FFTs on the rotation group," *J. Fourier Anal. and Appl.*, vol. 14, pp. 145–179, 2008.

[20] N. Jarosik, C. L. Bennett, J. Dunkley, B. Gold, M. R. Greason, M. Halpern, R. S. Hill, G. Hinshaw, A. Kogut, E. Komatsu, D. Larson, M. Limon, S. S. Meyer, M. R. Nolta, N. Odegard, L. Page, K. M. Smith, D. N. Spergel, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, "Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Sky maps, systematic errors, and basic results," *Astrophys. J.*, vol. 192, no. 2, pp. 1–14, 2011.