

ROBUST RECONSTRUCTION OF SPHERICAL SIGNALS WITH FINITE RATE OF INNOVATION

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ABSTRACT

We develop a robust method for the accurate reconstruction of non-bandlimited finite rate of innovation signals composed of finite number of Diracs. For the recovery of parameters of K Diracs defining the signal, the proposed method requires more than $(K + \sqrt{K})^2$ samples of the signal band-limited in harmonic domain such that the spherical harmonic transform can be computed using the samples. In comparison with the existing methods, the proposed method is robust in a sense that it does not require all Diracs to have distinct colatitude parameter. We first estimate the N number of Diracs which do not have distinct colatitude parameter. Once N is determined, the proposed method requires, at most, $\frac{N^2+N}{2} + 1$ unique and intelligently chosen rotations of the signal to recover all parameters accurately. We also provide illustrations to demonstrate the accurate reconstruction using the proposed method.

Index Terms— Finite rate of innovation, recovery of Diracs, spherical harmonics, non-bandlimited signals, unit sphere

1. INTRODUCTION

In many applications, signals are inherently defined on the sphere. These applications appear in wireless communication [1], cosmic microwave background [2], astrophysics [3], acoustics [4], planetary science [5], diffusion magnetic resonance imaging (dMRI) [6, 7]. To support accurate signal reconstruction and harmonic analysis in these application, many sampling schemes have been devised for band-limited signals (e.g., [8] and references therein) as these schemes enable the accurate computation of spherical harmonic transform (SHT), which is the well-known counterpart of the Fourier transform. However, these sampling schemes do not support accurate representation or reconstruction of non-bandlimited signals, such as, fiber orientations in diffusion weighted magnetic resonance imaging and microphone locations in spherical microphone array. In this work, we consider the problem of robust and accurate reconstruction of a class of non-bandlimited signals which have finite degree of freedom called the finite rate of innovation (FRI). Signals with FRI consist of finite number of Diracs distributed over the whole sphere.

In Euclidean domain, a sampling scheme has been proposed in [9] to sample signals with FRI based on the formulation of annihilating filter. This concept is adopted in [10] recently for the development of sampling scheme on the sphere to sample signals with FRI, where it has been demonstrated that the signals with FRI can be

accurately reconstructed, that is, parameters of Diracs can be recovered, by first band-limiting the signal and then taking finite number of samples over the grid defined by the sampling scheme that support accurate computation of SHT. The proposed method (algorithm) requires at least $4K^2$ spherical samples to reconstruct a stream of K Diracs, which has $3K$ degrees of freedom, on the sphere. More recently a sampling scheme has been proposed in [7] where the required number of spherical samples are reduced to $(K + \sqrt{K})^2$ samples for accurate recovery of the parameters of K Diracs on the sphere. However, the proposed method has a limitation that it requires that no two Diracs on the sphere share the same colatitude and no Diracs are placed on either of the poles ($\theta = 0$ and $\theta = \pi$, θ is formally defined in section 2.1).

In this work, we develop a method to accurately reconstruct a signal composed of K Diracs on the sphere using $(K + \sqrt{K})^2$ samples of the band-limited signal taken over optimal dimensionality sampling scheme [8]. The proposed method is based on intelligent choice of series of rotations to eliminate the limitation of the existing methods. Consequently, the proposed method is robust in a sense that it accurately recovers parameters of the signal and consequently enable accurate reconstruction. We first review the mathematical background and formulate the problem in Section 2. Existing method is reviewed in Section 3 and proposed developments are made in Section 4, where we also provide illustrations to corroborate the theoretical developments. Finally, concluding remarks are made in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Mathematical Background – Signals on the Sphere

The unit sphere (2-sphere) is defined as $\mathbb{S}^2 = \{\hat{\mathbf{u}} \in \mathbb{R}^3 : \|\hat{\mathbf{u}}\|_2 = 1\}$, where $\|\cdot\|_2$ denotes the Euclidean norm. The unit vector $\hat{\mathbf{u}}$ is parameterized in terms of colatitude angle $\theta \in [0, \pi]$ and longitude angle $\phi \in [0, 2\pi)$ as $\hat{\mathbf{u}} \equiv \hat{\mathbf{u}}(\theta, \phi) \triangleq (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)'$. Functions on the 2-sphere form a Hilbert space $L^2(\mathbb{S}^2)$ equipped with the inner product

$$\langle f, g \rangle \triangleq \int_{\mathbb{S}^2} f(\hat{\mathbf{u}}) \overline{g(\hat{\mathbf{u}})} ds(\hat{\mathbf{u}}), \quad (1)$$

between two functions f and g defined on \mathbb{S}^2 . Here $ds(\hat{\mathbf{u}}) = \sin \theta d\theta d\phi$ is the differential area element on \mathbb{S}^2 , $\overline{(\cdot)}$ denotes the complex conjugate and the integration is carried over the entire sphere. The inner product in (1) induces a norm $\|f\| \triangleq \langle f, f \rangle^{1/2}$. Functions with finite induced norm are defined as signals on the sphere.

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The spherical harmonic (SH) functions (or spherical harmonics), denoted by $Y_\ell^m(\hat{\mathbf{u}})$ for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$, form a complete set of basis functions for $L^2(\mathbb{S}^2)$ [11]. Here $|\cdot|$ denotes the absolute value operator. The SH coefficient $(f)_\ell^m$ of degree $\ell \geq 0$ and order $|m| < \ell$ is defined as $(f)_\ell^m = \langle f, Y_\ell^m \rangle$ [11]. The function f is said to be band-limited at degree L if $(f)_\ell^m = 0, \forall \ell \geq L, |m| \leq \ell$.

2.2. Problem under Consideration

We consider a signal as a weighted sum of K Diracs on the sphere given by

$$f(\hat{\mathbf{u}}) = \sum_{k=1}^K \alpha_k \delta(\hat{\mathbf{u}}, \hat{\mathbf{u}}_k), \quad (2)$$

where α_k is the complex weight and $\hat{\mathbf{u}}_k \equiv \hat{\mathbf{u}}(\theta_k, \phi_k)$ represents the location of k -th Dirac on the sphere. Here $\delta(\hat{\mathbf{u}}, \hat{\mathbf{u}}_k)$ is the spherical Dirac delta function which may be identified by its action on functions as $\langle f, \delta(\cdot, \hat{\mathbf{u}}_k) \rangle = f(\hat{\mathbf{u}}_k)$. The problem under consideration is to recover the parameters of the signal, that is complex weights α_k and locations $\hat{\mathbf{u}}_k$ for $k = 1, 2, \dots, K$ provided the samples of the signal f band-limited in harmonic domain.

In [7], an algorithm has been proposed for the recovery of parameters of f by band-limiting the signal at degree¹ $L = \lceil K + \sqrt{K} \rceil$, taking L^2 samples [8] of the band-limited signal and employing the annihilating filter method [9]. The proposed algorithm works accurately provided that the colatitude θ_k for each Dirac is distinct and $\theta_k \notin \{0, \pi\}$. To resolve this problem, a *random* rotation is applied to the coordinate system prior to reconstruction. However, this does not resolve the problem completely. There is a possibility that the rotated Diracs do not have distinct colatitude θ_k or $\theta_k \in \{0, \pi\}$. In this work, we resolve this problem and present a method to recover the parameters of the signal having Diracs with same colatitudes.

3. RECONSTRUCTION OF SIGNALS WITH FRI

Here, we review an algorithm presented [7] for the recovery of parameters of f . The proposed algorithm involves band-limiting of the signal at degree $L = \lceil K + \sqrt{K} \rceil$, taking L^2 samples [8] of the band-limited signal and employing the annihilating filter method [9].

We first take L^2 samples of the signal f band-limited at L using an optimal-dimensionality sampling scheme [8] which support the accurate computation of SH coefficients $(f)_\ell^m \forall \ell < L, |m| \leq \ell$ [8]. By employing sifting property of Dirac delta function, the representation of SH function $Y_\ell^m(\theta, \phi) = Y_\ell^m(\theta, 0)e^{im\phi}$ and noting that $Y_\ell^m(\theta, 0)$ is a product of $(\sin \theta)^{|m|}$ and a polynomial in $\cos \theta$ of degree $\ell - |m|$, we can express SH coefficient $(f)_\ell^m = \langle f, Y_\ell^m \rangle$ as

$$(f)_\ell^m = \sum_{p=0}^{\ell-|m|} c_{\ell m}^p d_{pm}, \quad (3)$$

where

$$d_{pm} = \sum_{k=1}^K (\alpha_k u_{km}) x_k^p \quad (4)$$

with $u_{km} = (\sin \theta_k)^{|m|} e^{-im\phi_k}$ and $x_k = \cos \theta_k$ and $c_{\ell m}^p$ denotes the coefficient associated with $(\cos \theta)^p$ of the polynomial defining

¹Here $\lceil \cdot \rceil$ denotes the integer ceiling function.

the SH $Y_\ell^m(\theta, 0)$. For a signal band-limited at L , there are $L - |m|$ SH coefficients of order m and degrees $|m| \leq \ell < L$, which can be used to recover d_{pm} for each m and $0 \leq p < L - |m|$ using inversion of (3).

Since d_{pm} is a linear combination of K powers of x_k , the annihilating filter technique [9] has been adopted in [7] to estimate $x_k = \cos \theta_k, k = 1, 2, \dots, K$. This estimation involves the construction of annihilating matrix \mathbf{Z} given by

$$\mathbf{Z} = \begin{bmatrix} d_{L-1,0} & d_{L-2,0} & \cdots & d_{L-K-1,0} \\ d_{L-2,0} & d_{L-3,0} & \cdots & d_{L-K-2,0} \\ \vdots & \vdots & \ddots & \vdots \\ d_{K,0} & d_{K-1,0} & \cdots & d_{0,0} \\ d_{L-2,1} & d_{L-3,1} & \cdots & d_{L-K-2,1} \\ d_{L-3,1} & d_{L-4,1} & \cdots & d_{L-K-3,1} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}, \quad (5)$$

followed by the computation of right singular vector \mathbf{v} of \mathbf{Z} and determination of $\theta_k, k = 1, 2, \dots, K$ by taking arccos of the roots of \mathbf{v} . As we are required to estimate K roots of \mathbf{v} , \mathbf{Z} needs to have at least K rows, which is ensured by the choice of band-limit taken as $L \geq K + \sqrt{K}$. We review the recovery of the longitude ϕ_k and amplitude α_k later in the paper.

4. ROBUSTNESS IN RECONSTRUCTION

The reconstruction of signals with finite rate of innovation presented in previous section is based on the assumption that the Diracs do not have the same colatitude. In practice, there is a possibility that the Diracs on the sphere defining the signal share same colatitude. Here we devise an algorithm for the recovery of signal parameters when Diracs have same colatitude.

We assume that there are N out of K Diracs which do not have unique colatitude parameter θ_k . In other words, the signal f , given in (2), has K Diracs placed on $K - N$ iso-latitude rings. Since N is not known in practice, it can be determined using following Lemma.

Lemma 1. *If the signal f consist of K Diracs, given in (2), has N non-unique colatitude parameter θ_k , the null-space of the annihilating matrix \mathbf{Z} , given in (5) is N dimensional.*

4.1. Determining the $K - N$ Iso-latitude Rings

As a consequence of Lemma 1, we do not get a unique solution to the annihilating matrix problem. However, we can determine unique θ_k for $k = 1, 2, \dots, K - N$ using \mathbf{Z} . Following Lemma 1, the matrix \mathbf{Z} has N dimensional null-space, denoted by $\mathcal{N}(\mathbf{Z})$. Using N vectors that span $\mathcal{N}(\mathbf{Z})$, we can determine unique colatitude parameters $\theta_k, k = 1, 2, \dots, K - N$ using the following Lemma.

Lemma 2. *If any vector v in the null space of the matrix \mathbf{Z} of rank $K - N$ represents the coefficients of polynomial of degree K , the polynomial associated with any vector \mathbf{v} has same $K - N$ roots.*

Proof. For any vector $v \in \mathcal{N}(\mathbf{Z})$, we have

$$\sum_{q=0}^K d_{p-q,m} v_q = 0, \quad (6)$$

for any $p = K, K + 1, \dots, L - 1$ and $|m| < L$. For N repeated colatitudes θ_k , the summation in the formulation of d_{pm} , given in

(4), includes $K - N$ terms, that is, we can express d_{pm} as

$$d_{pm} = \sum_{k=1}^{K-N} b_k x_k^p, \quad (7)$$

for unique $x_k = \cos \theta_k$, $k = 1, 2, \dots, K - N$, which upon substitution in (6) gives

$$\sum_{k=1}^{K-N} b_k x_k^p \underbrace{\sum_{q=0}^K v_q x_k^{-q}}_{V(x_k)} = 0, \quad (8)$$

which implies $V(x_k) = 0$ for each unique $x_k = \cos \theta_k$, $k = 1, 2, \dots, K - N$ and is equivalent to the statement of the theorem. \square

Since each polynomial associated with any vector in the N dimensional null space has same $K - N$ roots, we can uniquely determine $K - N$ colatitudes representing location of iso-latitude rings where all K Diracs are placed.

4.2. Using the Information of $K - N$ Iso-latitude Rings to Determine Rotation Parameters

Having information about $K - N$ unique iso-latitude rings, we here devise a method to determine the remaining N repeated colatitudes correctly. The K Diracs representing the signal f are distributed on these $K - N$ rings such that each ring contains at least one Dirac. Our method is based on rotating the signal *intelligently* such that the rotated signal do not have Diracs with same colatitude.

Using the known colatitudes, we want to estimate a rotation of the signal f which ensures that no two Diracs have the same colatitude θ_k after the rotation is applied. A possible solution is to rotate f around y -axis by β given by

$$2\beta < \min_{i,j=[1,K-N], i \neq j} (\theta_i, \pi - \theta_j, |\theta_i - \theta_j|), \quad (9)$$

which ensures that the rings do not overlap after the rotation is applied. This also ensures that there is no Dirac at either of the poles ($\theta_k = 0$ and $\theta_k = \pi$). This solution does not work if two Diracs residing on a single ring are symmetric with respect to x -axis, then even after the rotation around y -axis is applied there colatitude will be the same. Furthermore, the rotation around z -axis prior or after the rotation around y -axis may not eliminate the possibility of Diracs having same colatitude parameter. We here devise a method where we apply a series of rotations around z -axis prior to the rotation around y -axis to enable the accurate and robust determination of all K colatitudes. Let f_n be the signal obtained by rotating f around z -axis by $\gamma_n \in (0, \pi)$ and then rotating around y -axis by β given in (9). The SH coefficients of the rotated signal f_n can be obtained as [11]

$$(f_n)_\ell^m = \sum_{m'=-\ell}^{\ell} d_\ell^{m,m'}(\beta) e^{-im'\gamma_n} (f)_\ell^{m'}, \quad (10)$$

where $d_\ell^{m,m'}$ denotes the Wigner- d function of degree ℓ and orders m, m' [11].

Using the SH coefficients of the rotated signal, we employ (3) to construct \mathbf{Z} given in (5) for the rotated signal. We keep on applying

²We choose $\gamma_n \in (0, \pi)$ as rotation in this range can resolve all pair of Diracs with same colatitudes and longitudes that are symmetric with respect to x -axis.

rotations by choosing random, but unique, γ_n and β given by (9) until the rank of the matrix \mathbf{Z} is K . Once we determine such a rotation, all K colatitude parameters of the rotated signal can be computed accurately. We use the following Lemma to determine the total number of rotations required for the accurate recovery of all K colatitudes.

Lemma 3. *For a signal having K Diracs placed on $K - N$ isolatitude rings, we need to apply at most $\frac{N^2+N}{2} + 1$ different rotations on the signal to find a rotation for which the matrix \mathbf{Z} given in (5) has rank K .*

Proof. Since we can have $N + 1$ Diracs with same colatitude parameter in the worst case, there are at most $\binom{N+1}{2}$ pair of Diracs with longitude parameters symmetric around x -axis. Consequently, we need at most $\frac{N^2+N}{2} + 1$ rotations around z -axis followed by the rotation around y -axis by β given in (9) to obtain the matrix \mathbf{Z} of rank K . \square

4.3. Recovery of Parameters

Once the rotation parameters β and γ_n are determined such that the \mathbf{Z} given in (5) for the rotated signal has rank K , we recover K colatitude parameters, denoted by $\tilde{\theta}_k$, $k = 1, 2, \dots, K$ of the rotated signal, by first computing the right singular vector \mathbf{v} of \mathbf{Z} and then taking arccos of the roots of \mathbf{v} . For the recovery of longitudes, denoted by $\tilde{\phi}_k$ of the rotated signal and amplitudes α_k for $k = 1, 2, \dots, K$, we employ the formulation in (3) and (4) to express the SH coefficient of degree ℓ and orders $m = 0, 1$ as

$$(f_n)_\ell^0 = \sum_{k=1}^K \alpha_k \sum_{p=0}^{\ell} c_{\ell 0}^p (\cos \tilde{\theta}_k)^p, \quad (11)$$

$$(f_n)_\ell^1 = \sum_{k=1}^K \underbrace{\alpha_k \sin \theta_k e^{-i\phi_k}}_{g_k} \sum_{p=0}^{\ell} c_{\ell 0}^p (\cos \tilde{\theta}_k)^p, \quad (12)$$

which is computed using (10) for all $\ell < L$. Since there are $L > K + \sqrt{K}$ SH coefficient $(f_n)_\ell^0$ of order 0 and the colatitudes $\tilde{\theta}_k$ of Diracs for the rotated signal have been computed, we invert (11) to recover amplitudes α_k for all $k = 1, 2, \dots, K$. Furthermore, (12) is inverted to recover all g_k , $k = 1, 2, \dots, K$, which can be used to recover longitudes of the rotated signal as

$$\phi_k = -\text{Angle} \left(\frac{g_k}{\alpha_k} \right), \quad (13)$$

where $\text{Angle}(\cdot)$ returns the phase of the complex number. Now we have recovered the colatitudes $\tilde{\theta}_k$, longitudes $\tilde{\phi}_k$ and amplitudes α_k for all $k = 1, 2, \dots, K$. Using $\hat{\mathbf{w}}_k \triangleq \hat{\mathbf{w}}_k(\tilde{\theta}_k, \tilde{\phi}_k)$, which represents the location of the k -th Dirac of the rotated signal f_n , we can determine $\hat{\mathbf{u}}_k(\theta_k, \phi_k)$ as

$$\hat{\mathbf{u}}_k = \mathbf{R}^{-1} \hat{\mathbf{w}}_k, \quad (14)$$

where $\mathbf{R} \in \mathbb{R}^{3 \times 3}$ is the rotation matrix corresponding to the rotation operator that rotates the signal first around z -axis by γ_n and then around y -axis by β [11].

4.4. Illustrations

Here we provide examples to demonstrate the proposed method for the reconstruction of signals with FRI. We consider a signal f of the form given in (2) with $K = 14$ and $N = 5$, that is, we take 14 Diracs placed on 9 rings. We first take random parameters by choosing $\alpha_k, k = 1, 2, \dots, K$ with real and imaginary parts uniformly distributed in $[0, 1]$, $\theta_k, k = 1, 2, \dots, K - N$ uniformly distributed in $[0, \pi]$ and $\phi_k, k = 1, 2, \dots, K - N$ uniformly distributed in $[0, 2\pi)$. The remaining $N = 5$ Diracs are placed on first five rings symmetric around x -axis, that is, we choose $\theta_{K-N+k} = \theta_k$ and $\phi_{K-N+k} = 2\pi - \phi_k$ for $k = 1, 2, \dots, N$. We also add random $\omega_k \in [0, 2\pi)$ to each ϕ_k . Now we generate the SH coefficient $(f)_\ell^m$ for $\ell < K + \sqrt{K}$ and $|m| < \ell$ using (3). The signal obtained in spatial domain using these SH coefficients is shown in Fig. 1, which represents the signal f band-limited at $L = \lceil K + \sqrt{K} \rceil$. Using SH coefficients $(f)_\ell^m$, we recover the parameters using the proposed method. We analyse the rank of the matrix \mathbf{Z} is less than K for signal f and the rotated signal f_n obtained by choosing rotation parameter β given in (9) and $\gamma_n = \omega_n$ for $n = 1, 2, \dots, 6$. As expected, the rank is $K - 1$ for $n = 1, 2, \dots, 5$ and 14 for $n = 6$ when there are no Diracs in the signal having same colatitudes. For the case when \mathbf{Z} is of rank K , we recover the parameters using the proposed method with maximum error between the recovered parameters and original parameters on the order of 10^{-6} illustrating that the proposed method enables the accurate recovery of parameters. We also analyse the recovery of parameters of the signal with randomly placed $K = 3, 4, \dots, 14$ number of Diracs. We again randomly generate parameters and apply the proposed reconstruction method to recover the parameters. We repeat the experiment 10 times and compute the average value for the errors E_θ, E_ϕ and E_α between recovered and original colatitudes, longitudes and amplitudes respectively, which are plotted in Fig. 2 in dB scale, where it is evident that the proposed method allows sufficiently accurate recovery of parameters and consequently accurate reconstruction of signals with FRI. It can also be noted that the errors grow with the increase in the K number of Diracs. This is due to the decrease in the minimum spacing between Diracs with the increase in K that results in the ill-conditioning of the matrix \mathbf{Z} and consequently computation of colatitudes. More rigorous analysis of the performance of proposed method and the application to problems in acoustics and dMRI are subjects of future work.

5. CONCLUSIONS

We have proposed a method for robust and accurate recovery of parameters of the non-bandlimited signal with finite rate of innovation. The proposed method eliminates the limitation of the existing methods which require Diracs defining the signal to have distinct colatitude parameter. For a signal defined as a weighted sum of K Diracs, the proposed method first estimates N number of Diracs which do not have distinct colatitudes. Once N is estimated, we determine the colatitude parameter for $K - N$ Diracs, using which, we intelligently apply a sequence of $\frac{N^2 + N}{2} + 1$ rotations on the signal such that the colatitude parameter for the remaining K Diracs is estimated. Once colatitude parameter is determined for all Diracs, we recover the remaining parameters. We have also provided illustrations to support the developments and demonstrate that the proposed method enables accurate recovery of parameters.

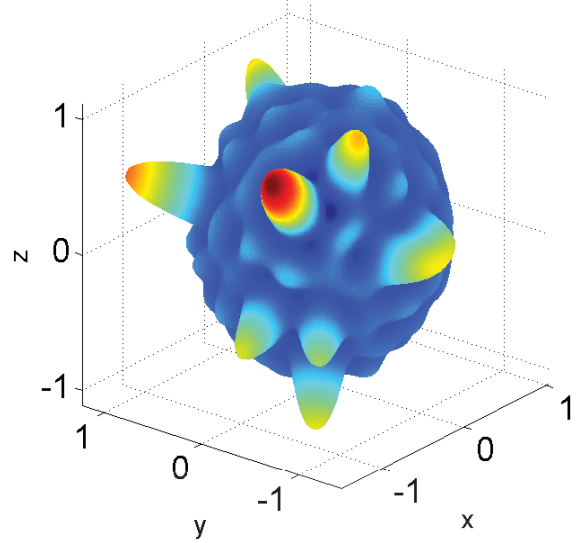


Fig. 1: The band-limited representation of the signal f obtained by first generating SH coefficients using (3) and then expanding the coefficients in SH basis.

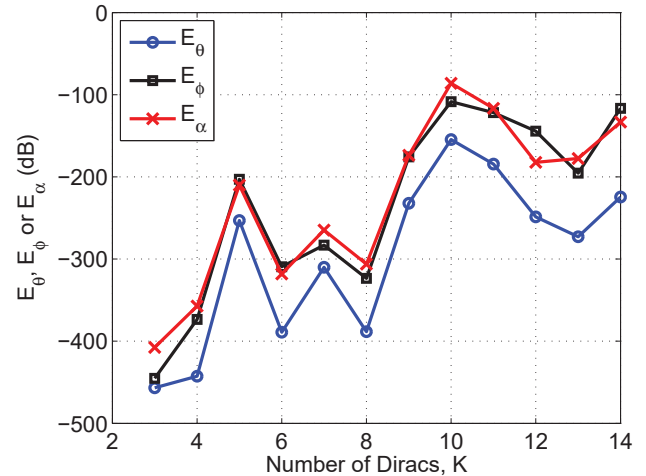


Fig. 2: Errors E_θ, E_ϕ and E_α between recovered and original colatitudes, longitudes and amplitudes respectively for different values of K (number of Diracs).

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