

# CONCENTRATION UNCERTAINTY PRINCIPLES FOR SIGNALS ON THE UNIT SPHERE

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## ABSTRACT

The uncertainty principle is an important and powerful tool, with many applications in signal processing. This paper presents two concentration uncertainty principles for signals on the sphere which relate the localization of the concentration of a signal in spatial and spectral domains, as an analogue of the general Donoho and Stark uncertainty principles in time-frequency analysis. Using the spherical and spectral truncation operators, we derive the  $L_1$ -norm and  $L_2$ -norm uncertainty principles which respectively relate the signal concentration in spatial and spectral domains as absolute value and the energy of a signal. We also analyze the sharpness of the bound imposed by the derived  $L_2$ -norm uncertainty principle. The proposed uncertainty measures can be applied to signal processing problems on the sphere.

**Keywords:** uncertainty principle, unit sphere, bandlimited signals, Slepian concentration problem, signal extrapolation.

## 1. INTRODUCTION

The uncertainty principle, originating from quantum mechanics, is an important and powerful tool in signal processing [1]. In time-frequency analysis, the classical (Heisenberg) uncertainty principle states that a function and its Fourier transform cannot be simultaneously well localized, i.e. they cannot be largely concentrated on intervals of small measure. A more general notion of uncertainty, herein referred to as concentration uncertainty principle, was provided by Donoho and Stark that a function and its Fourier transform cannot be largely concentrated on any set of small measure [2]. The authors in [2] defined two criteria for measuring concentrations as the absolute value of the signal ( $L_1$ -norm) and as the signal energy ( $L_2$ -norm). The uncertainty principle has many important applications in time-frequency analysis. The classical uncertainty principle provided the motivation for the famous Slepian's time-frequency concentration problem of optimally concentrating a signal in both spatial and spectral domains [3]. The concentration uncertainty principles have also been extensively applied in the problems of signal reconstruction and recovery [2], signal extrapolation [4] and compressive sampling [5].

Extending the uncertainty principle for signals defined on the unit sphere is an important problem in unit sphere signal processing. Signals defined on the unit sphere involve spherical harmonics as basis functions and have many applications in various branches of physical sciences and engineering [6, 7]. The fundamental problem under consideration here is to derive the concentration uncertainty principle that relates the simultaneous concentration of a signal in the spatial and spectral domains. The classical uncertainty principle

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is formulated for signals on the sphere in [8, 9] and has been applied in geodesy and geophysics to study the localization properties of wavelets and signals on the sphere [9–11]. The Slepian's concentration problem on the sphere has been investigated in [7, 10, 12] and the resulting eigenfunctions are analyzed using the classical uncertainty principle on the sphere that relates the variances of the signal in spatial and spectral domains. To the best of the authors' knowledge, the concentration uncertainty principles for signals on the sphere have not been formulated in the literature. Conceptually, the closed geometry of the sphere makes this task a non-trivial extension from the time-frequency case. For example, the sphere has a finite support in the spatial domain and is periodic in both co-latitude and longitude, which causes the spectral representation using spherical harmonic basis functions to be discrete. In addition, the intervals in the time domain correspond to the regions on the sphere which can vary in shape and may not necessarily be connected.

In this work, we derive two concentration uncertainty principles for signals on the sphere as an analog of the Donoho and Stark uncertainty principles in the time-frequency analysis. We formulate the selection operators in both spatial and spectral domains, which truncate a signal in the desired spatial or spectral region. We present the  $L_1$ -norm concentration uncertainty principle such that the absolute value of the signal is considered as concentration measure. Using the Hilbert-Schmidt norm of the selection operators, we also derive the  $L_2$ -norm concentration uncertainty principle where the concentration is measured as signal energy in the desired spatial and spectral regions. Finally, we analyze the sharpness of the proposed  $L_2$ -norm uncertainty principle bound by comparing it with the eigenvalue associated with the most concentrated bandlimited eigenfunction obtained from the Slepian's concentration problem on the sphere [10].

The rest of the paper is organized as follows. The mathematical background for the signals on the sphere is provided in Section 2. The formulation of the selection operators and the derivation and analysis of the uncertainty principles is presented in Section 3. Finally, Section 4 concludes the paper. Notations and terms:  $(\cdot)$  denotes the complex conjugate operation.  $\sup\{\cdot\}$  denotes supremum of a set,  $|\cdot|$  denotes the magnitude and  $\|(\cdot)\|_p$  denotes the  $L_p$  norm of the signal or operator which is defined later in Section 2.

## 2. MATHEMATICAL BACKGROUND

### 2.1. Signals on the Unit Sphere

We consider a function  $f(\theta, \phi)$  defined on the unit sphere  $\mathbb{S}^2 \triangleq \{\mathbf{r} \in \mathbb{R}^3 : \|\mathbf{r}\| = 1\}$ , i.e., if  $\mathbf{r} \in \mathbb{S}^2$  then  $\mathbf{r}$  is a unit vector,  $\theta \in [0, \pi]$  denotes the co-latitude measured with respect to the positive  $z$ -axis ( $\theta = 0$  corresponds to the north pole), and  $\phi \in [0, 2\pi)$  denotes the longitude and is measured with respect to the positive  $x$ -axis in the  $x - y$  plane. The inner product of two functions  $f, g \in L^2(\mathbb{S}^2)$  is

defined as

$$\langle f, g \rangle \triangleq \int_{\mathbb{S}^2} f(\Omega) \overline{g(\Omega)} d\Omega \quad (1)$$

where  $\Omega \equiv \{\theta, \phi\}$  parameterizes a point on the unit sphere and  $d\Omega = \sin \theta d\theta d\phi$ . The finite energy functions on the sphere such that  $\|f\| \triangleq \langle f, f \rangle^{1/2} < \infty$  are referred to as “signals on the unit sphere”. All such finite energy signals under inner product (1) form a complex Hilbert space  $L^2(\mathbb{S}^2)$ . In the following any reference to a signal is understood to be the same as “signal on the unit sphere”, unless otherwise stated.

For a signal  $f(\Omega)$ , its  $L_p$ -norm is defined as [13]

$$\|f\|_p = \left( \int_{\mathbb{S}^2} |f(\Omega)|^p d\Omega \right)^{1/p} \quad (2)$$

## 2.2. Spherical Harmonics

The spherical harmonics,  $Y_\ell^m(\theta, \phi)$ , for degree  $\ell \geq 0$  and order  $|m| \leq \ell$  are defined as [14]

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi} \quad (3)$$

where  $P_\ell^m$  are the associated Legendre polynomials defined for  $m \geq 0$  as

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} \sqrt{(1-x^2)^m} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \quad (4)$$

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x) \quad (5)$$

for  $|x| \leq 1$ . With the above definitions, spherical harmonic functions form an orthonormal set of basis signals for  $L^2(\mathbb{S}^2)$ , i.e., they satisfy  $\langle Y_\ell^m, Y_{\ell'}^{m'} \rangle = \delta_{\ell\ell'} \delta_{mm'}$ , where  $\delta_{ab}$  denotes the Kronecker delta and is equal to 1 for  $a = b$  and zero otherwise. By completeness [15], any signal  $f \in L^2(\mathbb{S}^2)$  can be expanded as

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\theta, \phi) \quad (6)$$

where equality is understood in terms of convergence in the mean, and

$$f_\ell^m \triangleq \langle f, Y_\ell^m \rangle = \int_{\mathbb{S}^2} f(\theta, \phi) \overline{Y_\ell^m(\theta, \phi)} d\Omega \quad (7)$$

are the spherical harmonic coefficients. In this work, we refer to the spherical harmonics domain, which consists of spherical harmonics coefficients of signal, as the spectral domain of a signal.

The Dirac delta function  $\delta(\Omega, \Omega')$  on the sphere has following expansion in spherical harmonics domain [7]

$$\delta(\Omega, \Omega') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\Omega) \overline{Y_\ell^m(\Omega')} \quad (8)$$

and obeys the following property

$$f(\Omega) = \int_{\mathbb{S}^2} \delta(\Omega, \Omega') f(\Omega') d\Omega' \quad (9)$$

We will also use the spherical harmonics addition theorem [15]

$$\sum_{m=-\ell}^{\ell} Y_\ell^m(\Omega) \overline{Y_\ell^m(\Omega')} = \frac{2\ell+1}{4\pi} P_\ell^0(\cos \Delta) \quad (10)$$

with  $\Omega = \{\theta, \phi\}$ ,  $\Omega' = \{\theta', \phi'\}$  and  $\cos \Delta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$ .

## 2.3. Operator Definition using Fredholm Integral Equation

Define an operator  $\mathcal{K}$  for signals on the sphere using general Fredholm integral equation [16]

$$(\mathcal{K}f)(\Omega) = \int_{\mathbb{S}^2} K(\Omega, \Omega') f(\Omega') d\Omega' \quad (11)$$

where  $K(\Omega, \Omega')$  is the kernel for an operator  $\mathcal{K}$ . Since it is important in the sequel, we define the  $L_p$ -norm of an operator  $\mathcal{K}$  as

$$\|\mathcal{K}\|_p = \sup_{f \in \mathbb{S}^2} \frac{\|\mathcal{K}f\|_p}{\|f\|_p} \quad (12)$$

Also, the Hilbert-Schmidt norm of an operator  $\mathcal{K}$  with kernel  $K(\Omega, \Omega')$  is given by [2]

$$\|\mathcal{K}\|_H = \left( \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} |K(\Omega, \Omega')|^2 d\Omega d\Omega' \right)^{1/2} \quad (13)$$

which is a bound on  $\|\mathcal{K}\|_2$ .

## 3. CONCENTRATION UNCERTAINTY PRINCIPLE

In this section, we formulate the concentration uncertainty principles for signals on the sphere such that the concentration is measured using the  $L_1$ -norm and  $L_2$ -norm of a signal respectively. More precisely, if the signal  $f(\Omega)$  on the sphere is concentrated on some spatial region  $R \subset \mathbb{S}^2$  and also concentrated on some spectral region  $N = [N_1, N_2]$  which denotes the spherical harmonic coefficients  $f_\ell^m$  with  $N_1 \leq \ell \leq N_2$  and  $-\ell \leq m \leq \ell$ , we develop the principles which relate the concentration measures and concentration regions in both spatial and spectral domains. Note that the region  $R$  does not need to be connected, whereas for the sake of simplicity, we consider the connected region  $N$  in spherical harmonics domain but this can also be generalized for non-connected regions. We first define the selection operators in both spatial and spectral domains which select the part of a signal in a selected spatial or spectral region [17]. Then we state and prove the uncertainty principles.

### 3.1. Selection Operators on the Sphere

**Definition 1** (Spatial Selection Operator). Define the spatial selection operator  $\mathcal{K}_R$  which selects the function in a region  $R$  with kernel  $K_R(\Omega, \Omega')$  as

$$K_R(\Omega, \Omega') \triangleq I_R(\Omega) \delta(\Omega, \Omega') \quad (14)$$

where  $I_R(\Omega) = 1$  for  $\Omega \in R \subset \mathbb{S}^2$  and  $I_R(\Omega) = 0$  for  $\Omega \in \mathbb{S}^2/R$  is an indicator function of the region  $R$ .

**Definition 2** (Spectral Selection Operator). Define the spectral selection operator  $\mathcal{K}_N$  with  $N = [N_1, N_2]$ , which selects the contribution of spherical harmonics in spectral region  $N$  in a signal and has the kernel  $K_N$  as

$$K_N(\Omega, \Omega') \triangleq \sum_{\ell=N_1}^{N_2} \sum_{m=-\ell}^{\ell} Y_\ell^m(\Omega) \overline{Y_\ell^m(\Omega')} \quad (15)$$

We note that both the spatial and spectral selection operators are idempotent and self-adjoint in nature. That is they are projection operators.

### 3.2. The $L_1$ -norm Uncertainty Principle

We first present the  $L_1$ -norm uncertainty principle which relates the concentration of signal in spatial and spectral domains to the measure of spatial and spectral regions, where the concentration is determined using the  $L_1$ -norm of a signal. We say that  $f$  is  $\epsilon_R$  concentrated in the spatial domain and  $\epsilon_N$  concentrated in the spectral domain if  $\|f - \mathcal{K}_R f\|_1 \leq \epsilon_R$  and  $\|f - \mathcal{K}_N f\|_1 \leq \epsilon_N$  respectively.

**Theorem 1** ( $L_1$ -norm Uncertainty Principle). If the unit  $L_1$ -norm signal  $f$  is  $\epsilon_R$  concentrated in the region  $R \subset \mathbb{S}^2$  and  $\epsilon_N$  concentrated in the spectral region  $N = [N_1, N_2]$ , then

$$\frac{A}{4\pi} \left( (N_2 + 1)^2 - N_1^2 \right) \geq \frac{1 - \epsilon_R - \epsilon_N}{1 + \epsilon_N} \quad (16)$$

where  $A = \int_{\mathbb{S}^2} I_R(\Omega) d\Omega$  denotes the area of the region  $R$ .

*Proof.* Since, the signal  $f$  is unit-norm, we have

$$\|f\|_1 = \int_{\mathbb{S}^2} |f(\Omega)| d\Omega = 1 \quad (17)$$

and using the given  $\|f - \mathcal{K}_R f\|_1 \leq \epsilon_R$  and  $\|f - \mathcal{K}_N f\|_1 \leq \epsilon_N$  and the fact that  $\|f - \mathcal{K}_R f\|_1 \geq \|f\|_1 - \|\mathcal{K}_R f\|_1$ , we obtain

$$\|\mathcal{K}_R f\|_1 + \epsilon_R \geq 1, \quad \|\mathcal{K}_N f\|_1 + \epsilon_N \geq 1 \quad (18)$$

Define a composite operator  $\mathcal{K}_{RN} = \mathcal{K}_R \mathcal{K}_N$ . From (17) and (18), we obtain the  $L_1$ -norm of this composite operator as

$$\|\mathcal{K}_{RN}\|_1 \geq \|\mathcal{K}_{RN} f\|_1 \geq 1 - \epsilon_R - \epsilon_N \quad (19)$$

Since, the operator  $\mathcal{K}_N$  is idempotent, using the spherical harmonic expansion of  $f$ , we can write

$$(\mathcal{K}_N f)(\Omega) = \sum_{\ell=N_1}^{N_2} \sum_{m=-\ell}^{\ell} \int_{\mathbb{S}^2} Y_{\ell}^m(\Omega) \overline{Y_{\ell}^m(\Omega')} (\mathcal{K}_N f)(\Omega') d\Omega' \quad (20)$$

Using the spherical harmonics addition theorem in (10), we can obtain from (20) that

$$|(\mathcal{K}_N f)(\Omega)| = \sum_{\ell=N_1}^{N_2} \frac{2\ell+1}{4\pi} \int_{\mathbb{S}^2} |P_{\ell}^0(\Delta)| \quad (21)$$

and we have  $\sup \|P_{\ell}^0(\Delta)\|_1 = 1$  and  $\|(\mathcal{K}_N f)\|_{\infty} = \max |(\mathcal{K}_N f)(\Omega)|$  for  $\Omega \in \mathbb{S}^2$ , which implies

$$\|(\mathcal{K}_N f)\|_{\infty} \leq \frac{1}{4\pi} \left( (N_2 + 1)^2 - N_1^2 \right) \|(\mathcal{K}_N f)\|_1 \quad (22)$$

For the composite operator  $\mathcal{K}_{RN}$ , we have

$$\|\mathcal{K}_{RN} f\|_1 = \int_R |(\mathcal{K}_N f)(\Omega)| d\Omega \leq \|\mathcal{K}_N f\|_{\infty} A \quad (23)$$

where  $A$  is the area of the region  $R$ . Using (23) and (22), we obtain

$$\|\mathcal{K}_{RN} f\|_1 \leq \frac{A}{4\pi} \left( (N_2 + 1)^2 - N_1^2 \right) \|(\mathcal{K}_N f)\|_1 \quad (24)$$

Combining (17), (18) and (19) with (24) gives the stated result.  $\square$

**Remark 1.** The factor  $\frac{A}{4\pi} \left( (N_2 + 1)^2 - N_1^2 \right)$  on the left hand side of (16) can be defined as a generalized space-bandwidth product, with the term  $A/4\pi$  being a measure of spatial region  $R$  and the term  $(N_2 + 1)^2 - N_1^2$  being a measure of spectral region  $N = [N_1, N_2]$ . For  $N_1 = 0$ , this space-bandwidth product is referred to as an equivalent of the Shannon number in [7, 10] for signals defined on the sphere.

### 3.3. The $L_2$ -norm Uncertainty Principle

Next, we present the uncertainty principle such that the concentration is measured using  $L_2$ -norm, which is a measure of energy of the signal and makes this principle more appealing and practical.

**Theorem 2** ( $L_2$ -norm Uncertainty Principle). If the unit  $L_2$ -norm signal  $f$  is  $\epsilon_R$  concentrated in the region  $R \subset \mathbb{S}^2$  such that  $\|f - \mathcal{K}_R f\|_2 \leq \epsilon_R$  and  $\epsilon_N$  concentrated in the spectral region  $N = [N_1, N_2]$  such that  $\|f - \mathcal{K}_N f\|_2 \leq \epsilon_N$ , then

$$\frac{A}{4\pi} \left( (N_2 + 1)^2 - N_1^2 \right) \geq (1 - \epsilon_R - \epsilon_N)^2 \quad (25)$$

where  $A = \int_{\mathbb{S}^2} I_R(\Omega) d\Omega$  denotes the area of the region  $R$ .

*Proof.* By definition,  $\|f\|_2 = 1$ ,  $\|f - \mathcal{K}_R f\|_2 \leq \epsilon_R$  and  $\|f - \mathcal{K}_N f\|_2 \leq \epsilon_N$ , which implies

$$\|\mathcal{K}_R f\|_2 \geq 1 - \epsilon_R, \quad \|\mathcal{K}_N f\|_2 \geq 1 - \epsilon_N \quad (26)$$

Define a composite operator  $\mathcal{K}_{NR} = \mathcal{K}_N \mathcal{K}_R$  composed of spatial selection operation  $\mathcal{R}$  followed by spectral selection  $\mathcal{K}_N$ . By substituting (14) and (15) in (11) and using the representation of Dirac delta in (8), we obtain the kernel  $K_{NR}(\Omega, \Omega')$  as

$$\begin{aligned} K_{NR}(\Omega, \Omega') &= \int_{\mathbb{S}^2} S_N(\Omega, \Omega'') S_R(\Omega'', \Omega) d\Omega'' \\ &= \sum_{\ell=N_1}^{N_2} \sum_{m=-\ell}^{\ell} Y_{\ell}^m(\Omega) \overline{Y_{\ell}^m(\Omega')} I_R(\Omega') \end{aligned} \quad (27)$$

The Hilbert Schmidt norm of the composite operator  $\mathcal{K}_{NR}$  can be obtained using (13) along with the spherical harmonics addition theorem in (10) and the fact that  $P_{\ell}^0(1) = 1$  as

$$\begin{aligned} \|\mathcal{K}_{NR}\|_H &= \left( \sum_{\ell=N_1}^{N_2} \frac{2\ell+1}{4\pi} \int_R d\Omega \right)^{1/2} \\ &= \left( \frac{A}{4\pi} \left( (N_2 + 1)^2 - N_1^2 \right) \right)^{1/2} \end{aligned} \quad (28)$$

Now, the effect of the composite operator on the concentration can be readily obtained from (26) as

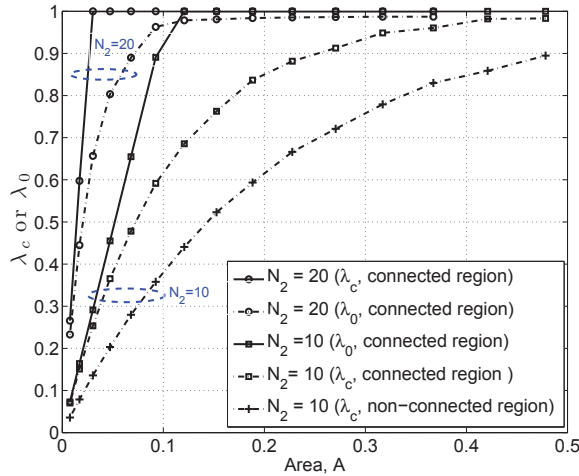
$$\|\mathcal{K}_{NR} f\|_2 \geq 1 - \epsilon_R - \epsilon_N \quad (29)$$

Using definition of the norm of operator in (12),  $\|\mathcal{K}_{NR}\|_2 \geq \|\mathcal{K}_{NR} f\|_2$  and the fact that  $\|\mathcal{K}_{NR}\|_H \geq \|\mathcal{K}_{NR}\|_2$  and combined with (28) and (29) gives the result in (25).  $\square$

**Remark 2.** We can infer from the proof of Theorem 2 that the composite operator  $\mathcal{K}_{RN} = \mathcal{K}_R \mathcal{K}_N$  is an adjoint of the operator  $\mathcal{K}_{NR}$  which implies that the Hilbert-Schmidt norm of these two composite operators is equal and given by (28).

### 3.4. Sharpness of the Uncertainty Principle Bound

We provide an analysis of the bound imposed by the  $L_2$ -norm uncertainty principle on the simultaneous spatial and spectral signal concentration, which is a measure of signal energy. We compare this bound with the largest eigenvalue obtained from the Slepian concentration problem for azimuthally symmetric signals on the sphere proposed in [7, 10, 12]. First, we consider the spatial polar cap regions  $R$  characterized by central angle  $\Theta$  with area  $A = \int_{\Theta}^{\pi} \int_0^{\Theta} \sin \theta d\theta d\phi =$



**Fig. 1.** Comparison of the uncertainty bound  $\lambda_0 = (N_2 + 1)^2(1 - \cos \Theta)/2$  obtained from (25) and the largest eigenvalue  $\lambda_c$  associated with the most concentrated bandlimited eigenfunction obtained from Slepian's concentration problem on the sphere [10]. Results are shown as a function of the area  $A$  of spatial polar cap regions with the consideration of different spectral regions  $N = [0, N_2]$  for  $N_2 = 10$  and  $N_2 = 20$ .

$2\pi(1 - \cos \Theta)$  and the spectral regions  $N = [0, N_2]$ . We use  $\lambda_c(\Theta, N_2)$  to denote the largest eigenvalue obtained numerically by solving the concentration problem on the sphere [10, 12], which finds the bandlimited signal of maximum spectral degree  $N_2$  with maximum energy concentration in the polar cap region of angle  $\Theta$ . We obtain the simplified form of the product on the left hand side in (25) for spatial and spectral regions under consideration as  $\lambda_0(\Theta, N_2) = (N_2 + 1)^2(1 - \cos \Theta)/2$ , which serves as an uncertainty bound. Fig 1 shows the comparison of  $\lambda_c(\Theta, N_2)$  and  $\lambda_0(\Theta, N_2)$  against  $\Theta$  for  $N_2 = 10$  and  $N_2 = 20$ , which indicates that  $\lambda_0$  serves as a sharp bound for the spatial polar cap region  $R$  and it gets tighter for smaller values of  $\lambda_0$ .

In the preceding analysis, we considered the connected polar cap region of central angle  $\Theta$ . Next, we consider a region of two non-connected polar caps centered at opposite poles ( $\theta = 0$  and  $\theta = \pi$ ) with the central angle of each polar cap being  $\cos^{-1}(1 + \cos \Theta)/2$ . It can be easily shown that the region of two polar caps have the same area  $A = 2\pi(1 - \cos \Theta)$ , thus it does not effect the uncertainty bound and it still holds but the tightness of the bound is comparatively reduced as illustrated by the 'non-connected region' curve in Fig 1. Summarizing our analysis, the bound or the limit imposed by concentration uncertainty principle is more sharper and tighter for connected regions and smaller values of space-bandwidth product respectively.

#### 4. CONCLUSIONS AND FUTURE WORK

In this work, we have investigated the concentration uncertainty principles for the signals on the sphere that relate the localization of concentration of a signal in spatial and spectral domains. Considering the concentration as absolute value of a signal, the  $L_1$ -norm uncertainty principle is derived. We also derived the more practical  $L_2$ -norm uncertainty principle using the Hilbert-Schmidt norm of the composite selection operator on the sphere. We showed that the uncertainty principle bound is relatively sharper and tighter for the connected regions as compared to non-connected regions. The

proposed uncertainty principles in this paper can be used to revisit the signal extrapolation problem on the sphere in the presence of noise [17]. They can also be applied to investigate compressive sampling on the sphere.

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