ACCURATE RECONSTRUCTION OF FINITE RATE OF INNOVATION SIGNALS ON THE SPHERE

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ABSTRACT

We propose a method for the accurate and robust reconstruction of the non-bandlimited finite rate of innovation signals on the sphere. For signals consisting of a finite number of Dirac functions on the sphere, we develop an annihilating filter based method for the accurate recovery of parameters of the Dirac functions using a finite number of observations of the bandlimited signal. In comparison to existing techniques, the proposed method enables more accurate reconstruction primarily due to the better conditioning of systems involved in the recovery of parameters. In order to reconstruct K Diracs on the sphere, the proposed method requires samples of the signal bandlimited in the spherical harmonic (SH) domain at SH degree equal or greater than $K + \sqrt{K + \frac{1}{4}} - \frac{1}{2}$. In comparison to the existing state-of-the-art technique, the required bandlimit, and consequently the number of samples, of the proposed method is (approximately) the same. We also conduct numerical experiments to demonstrate that the proposed technique is more accurate than the existing methods by a factor of 10^7 or more for $2 \leq K \leq 20$.

Index Terms— Unit sphere, sampling, finite rate of innovation, signal reconstruction, spherical harmonic transform

1. INTRODUCTION

Spherical signal processing techniques finds direct applications in diverse fields of science and engineering where signals are naturally defined on the sphere. Applications of these techniques include, but not limited to, spherical harmonic lighting in computer graphics [1], signal analysis in diffusion magnetic resonance imaging (dMRI) [2, 3], spectrum estimation in geophysics and cosmology [4], sound analysis and reproduction in acoustics [5] and placement of antennas in wireless communication [6]. To support signal analysis in these applications, accurate reconstruction of signals from a finite number of measurements is inherently required and is therefore of significant importance. In this paper, we consider the problem of sampling and accurate reconstruction of non-bandlimited finite rate of innovation (FRI) signal, consisting of finite K number of Dirac delta functions, on the sphere.

Many sampling schemes have been proposed in the literature (e.g., see [7] and references therein) for the acquisition of signals bandlimited in the spectral domain, which is enabled by the spherical harmonic (SH) transform – a natural counterpart of the Fourier transform for signals on the sphere [8]. For the accurate computation of SH transform and accurate reconstruction of a signal bandlimited at SH degree L (formally defined in Section 2.1), we require L^2 number of samples [7]. The sampling schemes for taking measurements of bandlimited signals, although permit accurate reconstruction of signal, are not suitable for sampling of non-bandlimited signals like an ensemble of spikes (Dirac delta functions in the limit) on the sphere which appear in applications in dMRI [3], acoustics and cosmology [9].

Based on the super-resolution theory [10], an algorithm has been developed in [11] for the reconstruction of FRI signals using semi-definite programming, root finding and least-squares. However, the method is iterative in nature and requires Dirac functions to satisfy a minimum separation condition. Recently, following the annihilating filter method devised for signals in one-dimensional Euclidean domain [12] and extended to 2D and higher dimensions [13], signal processing techniques have been proposed in [9, 14, 15] for the recovery of parameters of FRI signal on the sphere. The method proposed in [14] requires bandlimiting the FRI signal at $L = 2K$ for the reconstruction of K Diracs. To reduce the total number of measurements, an alternative reconstruction technique has been developed in [9] which requires the measurements of the FRI signal bandlimited at $L \geq (K + \sqrt{K})$ and therefore reduces the number of samples requirement by a factor of approximately four. For both of these methods, the error in the recovery of parameters increases with the number of Diracs due to ill-conditioning of the systems.

In this work, we also employ the annihilating filter method in order to develop a method for the accurate reconstruction of an FRI signal composed of K Diracs on the sphere. Our method bandlimits the signal at $L \geq (K + \sqrt{K + \frac{1}{4}} - \frac{1}{2})$ or $L \geq (K + \sqrt{K + \frac{5}{4}} + \frac{1}{2})$ and therefore takes (approximately) the same [1] number of samples compared to the best of existing methods. More importantly, in comparison to existing techniques, our method enables more accurate recovery of parameters of FRI signals as we demonstrate through numerical experiments. Before we present our proposed method in Section III and carry out its analysis in Section IV, we present the mathematical background and review of the existing methods for the problem under consideration.

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[1] Since bandlimit L is required to be an integer, the difference between the bandlimit required by the best of existing algorithms and the proposed method is zero or differs by one.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Mathematical Preliminaries – Signals on the Sphere

The 2-sphere or unit sphere is defined as $\mathbb{S}^2 = \{\hat{\boldsymbol{u}} \in \mathbb{R}^3 : |\hat{\boldsymbol{u}}|_2 = 1\}$, where we denote by $|\cdot|_2$ the Euclidean norm and by $\hat{\boldsymbol{u}}$ a unit vector in $\mathbb{R}^3$, parameterized in terms of θ and ϕ as $\hat{\boldsymbol{u}} \equiv \hat{\boldsymbol{u}}(\theta, \phi) \triangleq (\sin\theta\,\cos\phi,\ \sin\theta\,\sin\phi,\ \cos\theta)^{\mathsf{T}}$. Here $\theta \in [0, \pi]$ is the colatitude angle, $\phi \in [0, 2\pi)$ is the longitude angle and $(\cdot)^{\mathsf{T}}$ denotes the transpose operator. The inner product between two functions (or signals) g and h on the sphere is defined as

$$\langle g, h\rangle \triangleq \int_{\mathbb{S}^2} g(\hat{\boldsymbol{u}})\,\overline{h(\hat{\boldsymbol{u}})}\,ds(\hat{\boldsymbol{u}}), \tag{1}$$

where $ds(\hat{\boldsymbol{u}}) = \sin\theta\,d\theta\,d\phi$ is the differential area element on $\mathbb{S}^2$, $\overline{(\cdot)}$ denotes the complex conjugate and the integration is carried out over the entire sphere. The complex-valued functions on the 2-sphere form a Hilbert space $L^2(\mathbb{S}^2)$ equipped with the inner product defined in (1).

For the space $L^2(\mathbb{S}^2)$, spherical harmonic functions (or spherical harmonics for short), denoted by $Y_\ell^m(\hat{\boldsymbol{u}}) \equiv Y_\ell^m(\theta, \phi)$ for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$, serve as complete orthonormal basis [8]. Due to completeness of spherical harmonics, we can represent any signal $f \in L^2(\mathbb{S}^2)$ as

$$f(\hat{\boldsymbol{u}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (f)_\ell^m\, Y_\ell^m(\hat{\boldsymbol{u}}), \tag{2}$$

where $(f)_\ell^m \triangleq \langle f, Y_\ell^m\rangle$ [8] denotes the spherical harmonic coefficient of integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$. The spherical harmonic coefficients form the representation of a signal in spectral (Fourier) domain. We define the function f to be bandlimited in spectral domain at degree L if $(f)_\ell^m = 0,\ \forall\,\ell \geq L,\ -\ell \leq m \leq \ell$.

2.2. Problem Formulation

We consider an ensamble of K Diracs on the sphere given by

$$f(\hat{\boldsymbol{u}}) = \sum_{k=1}^{K} \alpha_k\, \delta(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}_k), \tag{3}$$

where $\hat{\boldsymbol{u}}_k \equiv \hat{\boldsymbol{u}}_k(\theta_k, \phi_k)$ represents the location of k-th Dirac on the 2-sphere and α_k is the complex amplitude. Here $\delta(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}_k)$ is the Dirac delta function defined on the sphere which, similar to its linear counterpart, is identified by its sifting property $\langle f, \delta(\cdot, \hat{\boldsymbol{u}}_k)\rangle = f(\hat{\boldsymbol{u}}_k)$. We want to accurately recover the amplitudes α_k and locations $\hat{\boldsymbol{u}}_k$ of K Diracs of f, given the samples of f bandlimited in the spectral domain.

2.3. Review of Existing Methods

For the recovery of parameters of signal of the form given in (3), an algorithm has been recently presented in [14], based on the annihilation filter method [12], for the recovery of the parameters of f which requires the computation of SH coefficients $(f)_\ell^m$ of f for degrees $\ell < 2K$ and orders $|m| \leq \ell$, which are computed by first convolving the signal f with a sampling kernel which bandlimits the signal at degree $L = 2K$. If the recently proposed optimal-dimensionality sampling [7] is employed for the computation of SH coefficients, the method requires L^2 samples of the signal f bandlimited at $L = 2K$. Employing the SH coefficients, the method then forms a Toeplitz system which enables the computation of ϕ_k using which α_k and θ_k are recovered. The method assumes that $\theta_k \notin \{0, \pi\}$ and $\theta_j \neq \pi - \theta_k$ when $\phi_j = \phi_k$ for $j, k = 1, 2, \ldots, K$ and $j \neq k$.

To reduce the number of samples required for the recovery of parameters, an algorithm has been presented more recently in [9], which requires spherical harmonic coefficients $(f)_\ell^m$ for $\ell < L$ and orders $|m| \leq \ell$ of the signal f bandlimited at SH degree[2] $L = \lceil K + \sqrt{K}\rceil$. Consequently, when compared to the method in [14], this method requires (approximately) *four* times less number of samples. The SH coefficients are then used to form an annihilating matrix [12] which enables the computation of θ_k, which are then used to recover the parameters α_k and ϕ_k. The algorithm works only when $\theta_k \notin \{0, \pi\}$ and $\theta_j \neq \theta_k$ for $j \neq k$ and $j, k = 1, 2, \ldots, K$. Although both of these methods allow recovery of parameters, the reconstruction error increases when the number of Diracs on the sphere increase as we illustrate later in the paper.

3. ACCURATE RECONSTRUCTION OF FRI SIGNALS

3.1. Formulation

By utilizing the sifting property of Dirac delta functions, we can express the spherical harmonic coefficient $(f)_\ell^m = \langle f, Y_\ell^m\rangle$ of f given in (3) as $(f)_\ell^m = \sum_{k=1}^{K} \alpha_k\, \overline{Y_\ell^m(\theta_k, \phi_k)}$. Note that $Y_\ell^m(\theta, \phi) = Y_\ell^m(\theta, 0)\, e^{im\phi}$ and $Y_\ell^m(\theta, 0)$ can be obtained by multiplying $(\sin\theta)^{|m|}$ with a polynomial in $\cos\theta$ of degree $(\ell - |m|)$ [8]. Therefore, denoting the coefficients of the polynomial in $\cos\theta$ by $c_{\ell m}^p$ (corresponding to $(\cos\theta_k)^p$), the SH coefficient $(f)_\ell^m$ can be expressed as

$$(f)_\ell^m = \sum_{k=1}^{K} \alpha_k \sum_{p=0}^{\ell-|m|} c_{\ell m}^p\,(\cos\theta_k)^p (\sin\theta_k)^{|m|} e^{-im\phi_k}, \tag{4}$$

For a fixed order m, rearranging (4) yields

$$(f)_\ell^m = \sum_{p=0}^{\ell-|m|} c_{\ell m}^p d_{pm}, \tag{5}$$

where

$$d_{pm} = \begin{cases} \sum_{k=1}^{K} (\alpha_k y_{kp}) x_k^m & 0 \leq m < L, \\ \sum_{k=1}^{K} (\alpha_k y_{kp}) \overline{x_k^{-m}} & -L < m < 0, \end{cases} \tag{6}$$

with $x_k = \sin\theta_k e^{-i\phi_k}$ and $y_{kp} = (\cos\theta_k)^p$. Clearly, both d_{pm} for $0 \leq m < L$ and $\overline{d_{pm}}$ for $-L < m < 0$ are linear combination of exponentials x_k^m and therefore are of special interest as the annihilating filter technique [12] can be used to recover x_k.

3.2. Annihilating Matrix Formulation

We consider that the measurements of the signal f bandlimited at degree L are available such that the spherical harmonic coefficients $(f)_\ell^m$ can be accurately computed for all degrees $\ell < L$ and all orders $|m| \leq \ell$. We shortly present the bandlimit L required for the accurate reconstruction of f.

In (5), $(f)_\ell^m$ for $|m| \leq \ell < L$ and d_{pm} for $0 \leq p < L - |m|$ form a linear system of equations for each $|m| < L$ with triangular coefficient matrix of size $(L-|m|) \times (L-|m|)$. Consequently, d_{pm} for $0 \leq p < L - |m|$ can be recovered *exactly* for each $|m| < L$ using (5).

[2] Here $\lceil\cdot\rceil$ denotes the integer ceiling function.

Once d_{pm} is computed, we employ the annihilating filter technique [12] to estimate x_k as d_{pm} is a linear combination of K powers of x_k. This technique is based on the construction of an annihilating matrix $\mathbf{Z}$ as follows

$$\mathbf{Z} = \begin{bmatrix} d_{0,L-1} & d_{0,L-2} & \cdots & d_{0,L-K-1} \\ d_{0,L-2} & d_{0,L-3} & \cdots & d_{0,L-K-2} \\ \vdots & \vdots & \ddots & \vdots \\ d_{0,K} & d_{0,K-1} & \cdots & d_{0,0} \\ \overline{d_{0,-(L-1)}} & \overline{d_{0,-(L-2)}} & \cdots & \overline{d_{0,-(L-K-1)}} \\ \overline{d_{0,-(L-2)}} & \overline{d_{0,-(L-3)}} & \cdots & \overline{d_{0,-(L-K-2)}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{d_{0,-(K)}} & \overline{d_{0,-(K-1)}} & \cdots & \overline{d_{0,0}} \\ d_{1,L-2} & d_{1,L-3} & \cdots & d_{1,L-K-2} \\ d_{1,L-3} & d_{1,L-4} & \cdots & d_{1,L-K-3} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,K} & d_{1,K-1} & \cdots & d_{1,0} \\ \overline{d_{1,-(L-2)}} & \overline{d_{1,-(L-3)}} & \cdots & \overline{d_{1,-(L-K-2)}} \\ \overline{d_{1,-(L-3)}} & \overline{d_{1,-(L-4)}} & \cdots & \overline{d_{1,-(L-K-3)}} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}, \quad (7)$$

and then computing its right singular vector $\mathbf{v}$.

Lemma 1. *Let f be a signal as defined in* (3)*. Assuming that $\alpha_k \in \mathbb{C}$ such that angle*[3] *$\angle\alpha_i \neq \angle\alpha_j$ for at least one (i, j) for $i, j = 1, 2, \ldots, K$, $i \neq j$, if the longitudes and colatitudes of K Diracs of f are such that $\theta_j \neq \pi - \theta_k$ when $\phi_j = \phi_k$ for $j, k = 1, 2, \ldots, K$, $j \neq k$, then the null-space $\mathcal{N}(\mathbf{Z})$ of the annihilating matrix $\mathbf{Z}$ with at least K rows is* 1*-dimensional.*

3.3. Recovery of Longitudes of Diracs

Now we employ the annihilating filter property to estimate x_k using $\mathbf{v} \in \mathcal{N}(\mathbf{Z})$. A finite impulse response (FIR) filter is known as annihilating filter if zeros of the filter are placed such that the filter annihilates the signal. Since $\mathbf{v} \in \mathcal{N}(\mathbf{Z})$, we have $\mathbf{d}_q^\mathsf{T}\mathbf{v} = 0$ for any $\mathbf{d}_q \triangleq \mathbf{Z}\{q, :\}$, that is, the q-th row of the annihilating matrix $\mathbf{Z}$. Consequently, $\mathbf{v}$ is a vector of coefficients of the FIR filter which annihilates the signal of the form (6). The transfer function of such annihilating filter is given by

$$V(z) \triangleq \prod_{k=1}^{K}(1 - x_k z^{-1}) \triangleq \sum_{n=0}^{K} v_n z^{-n}. \quad (8)$$

Since we have determined $\mathbf{v}$, we obtain the estimate of x_k for $1 \leq k \leq K$ by finding the K roots, denoted by $\tilde{x}_k$ for $1 \leq k \leq K$, of the annihilating filter. Using $\tilde{x}_k$ and noting that $x_k = \sin\theta_k e^{-i\phi_k}$, we recover the longitudes ϕ_k from $\tilde{x}_k$ as

$$\tilde{\phi}_k = -\angle\tilde{x}_k. \quad (9)$$

3.4. Bandlimit Requirement

For a signal f bandlimited at L, the maximum number of rows of $\mathbf{Z}$ which can be constructed is $2 \times (L-K) + 2 \times (L-K-1) + \cdots + 2 \times 2 + 2 \times 1$. Following Lemma 1, we require matrix $\mathbf{Z}$ to have at least K rows to ensure a unique vector $\mathbf{v} \in \mathcal{N}(\mathbf{Z})$. Consequently, we require

$$L \geq K + \sqrt{K + \frac{1}{4}} - \frac{1}{2}. \quad (10)$$

However, if $\alpha_k \in \mathbb{R}$ / $\alpha_k \in \mathbb{C} : \angle\alpha_i = \angle\alpha_j$ for all $i, j = 1, 2, \ldots, K$ then $\mathcal{N}(\mathbf{Z})$ in more than 1-dimensional. To reconstruct such a signal, we add a complex Dirac, with known location, to f to obtain a modified signal g consisting of $K + 1$ Diracs such that

$$(g)_\ell^m = (f)_\ell^m + \alpha_{K+1}\overline{Y_\ell^m}(\theta_{K+1}, \phi_{K+1}), \quad (11)$$

here we assume that $\angle\alpha_{K+1} \neq \angle\alpha_i$ for $i = 1, 2, \ldots, K$. We then reconstruct the annihilating matrix $\mathbf{Z}$ for the reconstruction of parameters of $K + 1$ Diracs using (11), (6) and (7) which, according to Lemma 1, results in a 1-dimensional $\mathcal{N}(\mathbf{Z})$. Since the signal to be reconstructed now consists of $K + 1$ Diracs, we need more number of samples of f to compute $(f)_\ell^m$ upto a greater bandlimit. The exact bandlimit requirement is therefore given by

$$L \geq \begin{cases} K + \sqrt{K + \frac{1}{4}} - \frac{1}{2} & \text{case 1}, \\ K + \sqrt{K + \frac{5}{4}} + \frac{1}{2} & \text{case 2}, \end{cases} \quad (12)$$

here case 1 refers to $\alpha_k \in \mathbb{C} : \angle\alpha_i \neq \angle\alpha_j$ for at least one (i, j), $i, j = 1, 2, \ldots, K$ and $i \neq j$ while case 2 refers to $\alpha_k \in \mathbb{R}$ / $\alpha_k \in \mathbb{C} : \angle\alpha_i = \angle\alpha_j$ for all $i, j = 1, 2, \ldots, K$. In the remaining sections, we define our algorithm for the reconstruction of f assuming that f satisfies Lemma 1. The extension of the algorithm to reconstruct the modified signal with $K + 1$ Diracs is straightforward.

3.5. Recovery of Colatitudes and Amplitudes of Diracs

Here we use the estimated $\tilde{x}_k$ to recover colatitude θ_k and amplitude α_k for $k = 1, \ldots, K$. For $p = 0$, we have d_{pm} for $m = 0, 1, \ldots, L-1$, which we explicitly rewrite, using (6), as

$$d_{0m} = \sum_{k=1}^{K} \alpha_k x_k^m. \quad (13)$$

Since we have chosen $L > K$, we can form the following Vandermonde system using (6):

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{K-1} & x_2^{K-1} & \cdots & x_k^{K-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_K \end{bmatrix} = \begin{bmatrix} d_{00} \\ d_{01} \\ \vdots \\ d_{0K-1} \end{bmatrix}. \quad (14)$$

Provided $x_k,\ k = 1, 2, \ldots, K$, are distinct as ensured by Lemma 1, the Vandermonde system above enables recovery of amplitudes $\tilde{\alpha}_k,\ k = 1, 2, \ldots, K$.

To recover colatitude parameter, we use d_{pm} for $p = 1$ and $m = 0, 1, \ldots, L-2$. Using d_{1m}, given in (6), and noting that $y_{k1} = \cos\theta_k$, we formulate another Vandermonde system given by

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{K-1} & x_2^{K-1} & \cdots & x_k^{K-1} \end{bmatrix} \begin{bmatrix} \alpha_1\cos\theta_1 \\ \alpha_2\cos\theta_2 \\ \vdots \\ \alpha_K\cos\theta_K \end{bmatrix} = \begin{bmatrix} d_{10} \\ d_{11} \\ \vdots \\ d_{1K-1} \end{bmatrix}.$$

The solution to above system yields an estimate of $\alpha_k\cos\theta_k$, denoted by $\text{est}\{\alpha_k\cos\theta_k\}$, for $k = 1, 2, \ldots, K$. Since we have already recovered amplitude as $\tilde{\alpha}_k$, we recover the colatitude as

$$\tilde{\theta}_k = \arccos\left[\frac{\text{est}\{\alpha_k\cos\theta_k\}}{\tilde{\alpha}_k}\right],\ k = 1, 2, \ldots, K. \quad (15)$$

[3] $\angle(\cdot)$ denotes the phase of the complex argument.

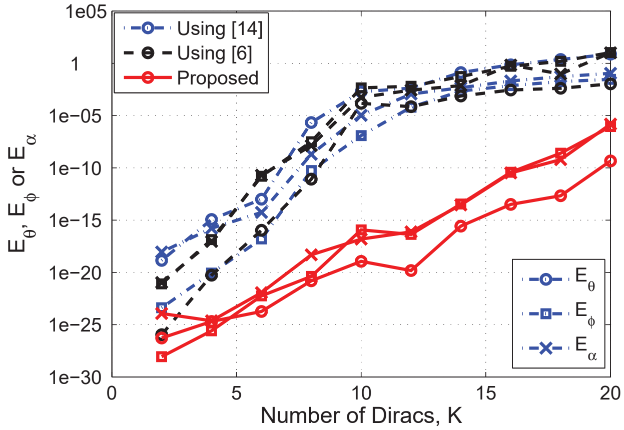

Fig. 1: Mean Error E_α, E_ϕ and E_θ between recovered and original amplitudes, longitudes and colatitudes respectively for different values of $2 \leq K \leq 20$ (number of Diracs).

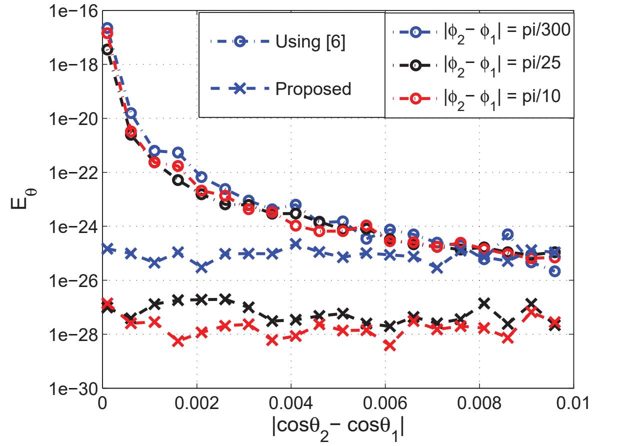

Fig. 2: Mean Error E_θ between recovered and original colatitudes for $K = 2$ at different locations of the two Diracs on the sphere.

4. ANALYSIS

4.1. Bandlimit Requirement Analysis

As mentioned earlier, we require L^2 number of measurements of the signal bandlimited at L to compute its spherical harmonic coefficients. Consequently, it is desirable for a method to have smaller bandlimit requirements to reduce the number of measurements. For the recovery of parameters of the signal consisting of K Diracs on the sphere, the bandlimit required by the proposed algorithm is either $L = \lceil K + \sqrt{K + \frac{1}{4}} - \frac{1}{2} \rceil$ or $L = \lceil K + \sqrt{K + \frac{5}{4}} + \frac{1}{2} \rceil$, which is much smaller as compared to $L = 2K$ required by the method in [14] and the same (*or* 1 *less or* 1 *more respectively*) than $L = \lceil K + \sqrt{K} \rceil$ required for the algorithm presented in [9].

4.2. Accuracy Analysis

In order to compare the recovery/reconstruction error of the proposed method with the algorithms presented in [14] and [9], we implement each method in MATLAB and recover the parameters of the signal f given by (3), by conducting following experiment. For each $K = 2, 4, \ldots, 20$, we randomly choose the parameters[4] $\theta_k \in (0, \pi)$, $\phi_k \in [0, 2\pi)$ and α_k with real and imaginary parts taken from a uniform distribution in the interval $[-1, 1]$ for $k = 1, 2, \ldots, K$. For each method and each K, we recover the parameters $\tilde{\theta}_k$, $\tilde{\phi}_k$ and $\tilde{\alpha}_k$ of the signal and compute the mean-squared errors given by $E_\theta = \frac{1}{K}\sum_{k=1}^{K} |\tilde{\theta}_k - \theta_k|^2$, $E_\phi = \frac{1}{K}\sum_{k=1}^{K} |\tilde{\phi}_k - \phi_k|^2$, $E_\alpha = \frac{1}{K}\sum_{k=1}^{K} |\tilde{\alpha}_k - \alpha_k|^2$. We plot these errors, averaged over 1000 trials of the experiment, in Fig. 1, where it is evident that the proposed method enables more accurate recovery of parameters when compared to other methods in literature. On average, the proposed algorithm outperforms the other methods in terms of smaller recovery error by a factor up to 10^7.

[4]The parameters are randomly generated such that we have K distinct θ_k and $x_k = \sin\theta_k e^{-i\phi_k}$ as the method in [9] requires θ_k, $k = 1, 2, \ldots, K$ to be unique, whereas the method in [14] and our proposed method require x_k, $k = 1, 2, \ldots, K$ to be unique (Lemma 1). This is avoided by imposing a condition that the K Diracs have at least $\frac{\pi}{3K}$ angular distance among them.

4.3. Better Conditioning of Our System

In this section we see why the proposed method is more accurate than [9, 14] by analyzing the relation between the reconstruction error and the minimum distance among the roots x_k of the annihilating filter. We consider a complex signal consisting of two Diracs on the Sphere. The first Dirac is located arbitrarily at $\hat{\boldsymbol{u}}_1(\theta_1, \phi_1) = \hat{\boldsymbol{u}}_1(\pi/2, \pi/6)$. We consider three different values of ϕ_2, such that $|\phi_2 - \phi_1| = \{\frac{\pi}{10}, \frac{\pi}{25}, \frac{\pi}{300}\}$. For each value of ϕ_2, θ_2 is choosen such that $|\cos\theta_2 - \cos\theta_1|$ varies from 1×10^{-2} to 1×10^{-4} in steps of 5×10^{-4}. For each case, we recover the colatitudes $\{\theta_1, \theta_2\}$ using [9] and the proposed algorithm separately. In [9], $x_k = \cos\theta_k$ is real and 1-dimensional. Therefore, we see from Fig. 2 that as $|\cos\theta_2 - \cos\theta_1| = |x_2 - x_1|$ is reduced, the reconstruction error is increased significantly irrespective of the value of ϕ_2. However, in the proposed algorithm, $x_k = \sin\theta_k e^{-i\phi_k}$ is complex and 2-dimensional. Therefore, as we reduce $|\cos\theta_2 - \cos\theta_1|$, the reconstruction error is not increased because $|x_2 - x_1|$ does not only depend on $|\cos\theta_2 - \cos\theta_1|$. Rather the reconstruction error depends on $|\phi_2 - \phi_1|$ as well and we can get smaller reconstruction error by increasing $|\phi_2 - \phi_1|$ as well. For each synthetic signal, we repeated the reconstruction 1000 times using [9] and the proposed algorithm to compute the average reconstruction error E_θ plotted in Fig. 2.
The proposed method is also more accurate than [14]. Although both [14] and the proposed method have 2-dimensional roots x_k of the annihilating filter, our proposed method uses d_{pm} for all possible values of p to construct the annihilating matrix while [14] uses only $p = 0$ to construct the annihilating matrix.

5. CONCLUSIONS

In this work, we have proposed a method for accurate reconstruction of an FRI signal consisting of K Dirac functions on the sphere. The proposed method takes samples of the signal bandlimited in the SH domain at the SH degree $L = \lceil K + \sqrt{K + \frac{1}{4}} - \frac{1}{2} \rceil$ or $L = \lceil K + \sqrt{K + \frac{5}{4}} + \frac{1}{2} \rceil$ for the computation of SH transform of the signal f. Following the computation of SH coefficients, we recover the parameters of the Diracs using the annihilating filter method, root finding and solving a series of linear systems. The proposed method requires (approximately) the same number of samples compared to the best of existing methods. More importantly, the error in the recovery of parameters is significantly smaller.

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