

Efficient Computation of Commutative Anisotropic Convolution on the 2-Sphere

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Abstract—Recently, the commutative anisotropic convolution has been defined for signals defined on the 2-sphere. Here, we present exact and efficient methods for computation of commutative convolution of two signals defined on the sphere. For fast computation of commutative convolution, we first review the use of existing efficient techniques developed to evaluate SO(3) convolution. By employing the factoring of a rotation into two rotations, followed by the separation of variables, we propose a fast algorithm for the efficient computation of commutative convolution. In terms of computational complexity, our proposed algorithm provides a saving of $O(N)$ over the existing algorithms, where the convolution output is evaluated on $O(N^2)$ samples on the 2-sphere. Through numerical experiments, we also verify the improvement in the computational complexity.

Index Terms—convolution, 2-sphere (unit sphere), spherical harmonics.

I. INTRODUCTION

Analysis and processing of signals defined on the sphere have direct applications in various branches of science and engineering such as geophysics [1], astrophysics [2], computer graphics [3], electromagnetic inverse problems [4] and wireless channel modeling [5]. Signal processing techniques have been extended from Euclidean domain to the spherical domain to deal with the signals defined on the sphere (e.g., [6]–[10]). Among these developments, the convolution of two signals on defined on the sphere is one of the basic signal processing tools.

There exist various formulations of convolution on the sphere (e.g., [6], [11], [12]), which do not well emulate the convolution definition in the Euclidean domain. The convolution definition in [6] takes into account the rotation of a signal by all independent three Euler angles in order to keep the convolution output on the sphere, which results in an extra averaging over an azimuthal rotation.

Due to this extra averaging, this convolution becomes equivalent to an isotropic convolution [7] and therefore the anisotropic or directional features of the rotated signal do not contribute towards the convolution output. Furthermore, due to an extra averaging over azimuthal rotation, this convolution definition is not commutative in general. In order to incorporate the directional features of the rotated signal, the convolution has been defined in [11] as anisotropic convolution, which involves the rotation by all Euler angles, due to which the output of the convolution is not defined on the 2-sphere, instead it is defined on SO(3). We refer this convolution as SO(3) convolution. We also note that SO(3) convolution is not commutative either.

Recently, a convolution definition on the 2-sphere has been proposed in [9] which simultaneously satisfies the following properties: 1) it is anisotropic, 2) the output is defined on 2-sphere and 3) it is commutative. This convolution has been referred as commutative anisotropic convolution. In philosophy, the commutative anisotropic convolution has been formulated by imposing a controlled dependency between the two rotations around z -axis involved in the SO(3) convolution [11]. Such controlled dependency incorporates the anisotropy, yields the commutativity and keeps the output of convolution on 2-sphere.

In this work, we consider the problem to efficiently evaluate the commutative anisotropic convolution. Since the commutative anisotropic convolution [9] can be considered as the mapping of SO(3) convolution with a constraint that the two rotations around z -axis depend on each other, the commutative anisotropic convolution [9] can be evaluated by employing the existing fast techniques for the computation of SO(3) convolution [11]. We show that evaluation of commutative anisotropic convolution using existing SO(3) convolution involves some computational redundancy and the com-

mutative anisotropic convolution can be evaluated more efficiently.

Here, we develop a method to efficiently compute the commutative anisotropic convolution. We present a range of algorithms from the direct quadrature evaluation, to a semi-fast algorithm that employs the existing efficient methods for SO(3) convolution, to the proposed fast algorithm. We show that the proposed fast algorithm is computationally efficient than the direct quadrature evaluation and semi-fast algorithm. Later, through numerical experiments, we verify the computational complexity of the proposed fast algorithm.

The rest of the paper is structured as follows. We provide mathematical preliminaries for signals on the 2-sphere, and briefly review the definitions of convolution on the sphere in Section II. We present different algorithms for the computation of commutative convolution in Section III. Simulation results are presented in Section IV and Section V concludes the paper.

II. MATHEMATICAL BACKGROUND AND PROBLEM FORMULATION

A. Mathematical Preliminaries

1) *Signals on the Sphere*: We consider the complex valued square integrable functions $f(\theta, \phi)$ defined on the 2-sphere $\mathbb{S}^2 \triangleq \{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r}| = 1\}$. Here (θ, ϕ) parameterize a point on unit sphere, where $\theta \in [0, \pi]$ denotes the colatitude measured from the positive z -axis and $\phi \in [0, 2\pi)$ denotes the longitude from the positive x -axis in the $x-y$ plane. We also use ϑ and φ to denote colatitude and longitude, respectively.

Define the inner product of two functions $f(\theta, \phi)$ and $h(\theta, \phi)$ defined on the 2-sphere as

$$\langle f, h \rangle \triangleq \int_{\mathbb{S}^2} f(\theta, \phi) \overline{h(\theta, \phi)} \sin \theta d\theta d\phi, \quad (1)$$

where $\sin \theta d\theta d\phi$ denotes the differential area element and the integration is carried out over the 2-sphere. The functions of the form $f(\theta, \phi)$ with the inner product defined in (1) form Hilbert space $L^2(\mathbb{S}^2)$. The norm $\|f\| \triangleq \langle f, f \rangle^{1/2}$ is induced by the inner product in (1). We refer the functions with finite induced norm as signals on the sphere.

2) *Spherical Harmonics*: The Hilbert space $L^2(\mathbb{S}^2)$ is separable and spherical harmonics are archetype complete orthonormal set of basis functions. The spherical harmonic functions $Y_\ell^m(\theta, \phi)$ are defined for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$ as

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}, \quad (2)$$

where P_ℓ^m denotes the associated Legendre function [13]. By completeness of spherical harmonics, any signal $f \in L^2(\mathbb{S}^2)$ can be expressed as

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_\ell^m Y_\ell^m(\theta, \phi) \quad (3)$$

where $(f)_\ell^m$ denotes the spherical harmonic coefficient of degree ℓ and order m and is given by

$$(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle = \int_{\mathbb{S}^2} f(\theta, \phi) \overline{Y_\ell^m(\theta, \phi)} \sin \theta d\theta d\phi \quad (4)$$

Also, the signal is defined to be band-limited function on the sphere with maximum spherical harmonic degree or band-limit L_f if $(f)_\ell^m = 0, \forall \ell > L_f$.

3) *Rotation of a Signal on the 2-Sphere*: Rotation of a signal defined on the sphere is often parametrized in terms of Euler angles [13], [14]. We define the Euler angles $(\varphi, \vartheta, \omega) \in \text{SO}(3)$, where $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ and $\omega \in [0, 2\pi)$. Also define the rotation operator $\mathcal{D}(\varphi, \vartheta, \omega)$, which rotates a function on the sphere in a sequence of ω rotation about z -axis, then ϑ rotation about y -axis followed by the φ rotation about z -axis. We also note the effect of rotation operator on spherical harmonic coefficient of the signal given by [13], [14]

$$\begin{aligned} (\mathcal{D}(\varphi, \vartheta, \omega)f)_\ell^m &\triangleq \langle \mathcal{D}(\varphi, \vartheta, \omega)f, Y_\ell^m \rangle \\ &= \sum_{m'=-\ell}^{\ell} D_\ell^{m, m'}(\varphi, \vartheta, \omega) (f)_{\ell}^{m'}, \end{aligned} \quad (5)$$

where $D_\ell^{m, m'}(\varphi, \vartheta, \omega)$ are the Wigner- D functions given by [14]

$$D_\ell^{m, m'}(\varphi, \vartheta, \omega) = e^{-im\varphi} d_\ell^{m, m'}(\vartheta) e^{-im'\omega}, \quad (6)$$

and $d_\ell^{m, m'}(\vartheta)$ denotes the Wigner- d function [14].

B. Convolution on the 2-Sphere

There exist various formulations of convolution on the sphere in the literature [6], [11], which do not well emulate the definition of convolution in the Euclidean domain. Here, we first review the definition of anisotropic convolution (SO(3) convolution) [11]. Later, we present the commutative anisotropic convolution [9], which has been recently proposed and serve as a close analog of its counterpart in Euclidean domain. The following definition has been referred as anisotropic convolution in the literature [11]

$$\begin{aligned} g(\varphi, \vartheta, \omega) &= h \star f \\ &\triangleq \int_{\mathbb{S}^2} (\mathcal{D}(\varphi, \vartheta, \omega)h)(\theta, \phi) f(\theta, \phi) \sin \theta d\theta d\phi. \end{aligned} \quad (7)$$

Since the rotation operator is parameterized in terms of three independent rotations, the domain of the output of the convolution is $\text{SO}(3)$, instead of \mathbb{S}^2 . We refer this convolution definition as $\text{SO}(3)$ convolution. We refer $f(\theta, \phi)$ as signal and $h(\theta, \phi)$ as filter, but they can be any complex-valued signals on the sphere. Another definition of convolution appears in [6] with output defined on \mathbb{S}^2 , but with a consideration that the filter is isotropic. We note that none of these definitions are commutative in general.

Recently, commutative and anisotropic convolution on the sphere has been proposed, which accommodates anisotropic filter, satisfies the commutativity property and produces the output defined on the 2-sphere (\mathbb{S}^2).

Definition 1 (Commutative Anisotropic Convolution): With the consideration that the convolution on the sphere satisfies commutativity and only two independent rotations should be involved in the convolution on the sphere, the anisotropic and commutative convolution has been defined in [9]

$$g_c(\vartheta, \varphi) = (f \odot h)(\vartheta, \varphi) \triangleq \int_{\mathbb{S}^2} (\mathcal{D}(\varphi, \vartheta, \pi - \varphi)h)(\theta, \phi) f(\theta, \phi) \sin \theta d\theta d\phi, \quad (8)$$

with the convolution output defined on \mathbb{S}^2 . Also, it can be verified that this definition is commutative, $(f \odot h)(\vartheta, \varphi) = (h \odot f)(\vartheta, \varphi)$.

This convolution definition can be interpreted as a special case of $\text{SO}(3)$ convolution in (7), where the rotation ω is now constrained to be a function of φ , that is, $\omega = \pi - \varphi$, which is a necessary and sufficient condition in order to preserve the convolution commutativity (Theorem 1 in [9]). Furthermore, following the spectral analysis of the commutative anisotropic convolution given in [9], we note that the convolution output $g_c(\vartheta, \varphi)$ given in (8) is not a band-limited function on the sphere, even when both signal $f(\theta, \phi)$ and filter $h(\theta, \phi)$ are band-limited functions.

C. Problem Statement

In this work, we consider the problem to efficiently evaluate the commutative anisotropic convolution in (8). Since the commutative anisotropic convolution is a special case of $\text{SO}(3)$ convolution, it can be computed efficiently by employing the existing efficient algorithms [11] for $\text{SO}(3)$ convolution to compute $g(\varphi, \vartheta, \omega)$ in (7) and then mapping $g(\varphi, \vartheta, \omega)$ to $g_c(\varphi, \vartheta)$ in (8) whose domain is \mathbb{S}^2 with a constrain $\omega = \pi - \varphi$. However, this method of computation of $g(\varphi, \vartheta)$ involves

redundancy as we do not need to compute the $\text{SO}(3)$ convolution for all values of ω , instead, we need to compute for only $\omega = \pi - \varphi$. In order to avoid this redundancy, we present fast algorithm for the efficient computation of commutative anisotropic convolution. In terms of computational complexity, our proposed fast algorithm provides a saving of $O(N)$ over existing efficient techniques, when the convolution output is defined on $O(N^2)$ number of samples on the sphere.

III. EFFICIENT COMPUTATION OF COMMUTATIVE ANISOTROPIC CONVOLUTION

For the computation of commutative anisotropic convolution given in (8), we present here a range of algorithms, from the direct quadrature evaluation, to the semi-fast algorithm that employs the existing efficient methods for $\text{SO}(3)$ convolution, to the proposed fast algorithm where we use the factoring of rotation approach [15] and employ FFT for efficient computation. We show that the proposed fast algorithm is computationally efficient than the direct quadrature evaluation and semi-fast algorithm. Later, through numerical experiments, we verify the computational complexity of the proposed fast algorithm.

A. Discretization of \mathbb{S}^2 and $\text{SO}(3)$

For representation of a signal on the sphere, it is required to define discretization of both the spherical coordinates of the unit sphere and the Euler angle representation of $\text{SO}(3)$. We consider the equiangular sampling scheme tessellation schemes, which support the computation of exact quadrature for band-limited signals.

For the unit sphere domain, we use the equiangular sampling scheme [16] $\mathfrak{S}(L) = \{\theta_{n_\theta} = 2\pi n_\theta / (2L + 1), \phi_{n_\phi} = 2\pi n_\phi / (2L + 1) : 0 \leq n_\theta \leq L, 0 \leq n_\phi \leq 2L\}$ as a grid of $(L + 1) \times (2L + 1)$ sample points on the sphere. Using the quadrature weights derived in [16], the integral of a band-limited function f with band-limit L_f over the sphere can be computed exactly as quadrature using the sampling scheme $\mathfrak{S}(L_f)$. Since the commutative convolution output $g_c(\vartheta, \varphi)$ given in (8) is not a band-limited function, therefore we cannot associate the size of the sampling grid for output with the band-limit of the output. The larger the size of the grid, the better is the resolution of the convolution output. We use N to characterize the sampling grid $\mathfrak{S}(N)$ for the commutative anisotropic convolution output.

For the discretization of Euler angle representation of $\text{SO}(3)$, we consider equiangular tessellation scheme $\mathfrak{C}(L) = \{\varphi_{n_\varphi} = 2\pi n_\varphi / (2L + 1), \vartheta_{n_\vartheta} = 2\pi n_\vartheta / (2L +$

1), $\omega_{n_\omega} = \pi(2n_\omega + 1)/(2L + 1) : 0 \leq n_\vartheta \leq L, 0 \leq n_\varphi, n_\omega \leq 2L\}$ as a grid of $(2L + 1) \times (L + 1) \times (2L + 1)$ sample points.

B. Direct Quadrature Evaluation

Here, we discuss computation of the commutative convolution by evaluating the integral in (8) directly using quadrature rule on the sphere. If the output is required to be computed on the equiangular grid $\mathfrak{S}(N)$ on the sphere, the signals f and h are required to be sampled on the same grid. By virtue of sampling theorem for band-limited signals on the sphere [16], if $N \geq \max(L_f, L_h)$, the integral can be computed *exactly* as summation over the spatial domain samples by employing the quadrature weights associated with the sampling scheme. Since the integral is computed as two dimensional summation over the grid $\mathfrak{S}(N)$, evaluated for each sample point on the two dimensional grid $\mathfrak{S}(N)$ of the output, the computational complexity to obtain the convolution output $g_c(\vartheta, \varphi)$ in (8) on the grid $\mathfrak{S}(N)$ is $O(N^4)$.

C. Semi-Fast Algorithm - SO(3) Convolution Case

Here, we present the computation of commutative convolution using the fast algorithm for SO(3) convolution described in [11] to obtain the convolution output $g(\varphi, \vartheta, \omega)$ given in (7) and defined on SO(3), followed by the mapping along $\omega = \pi - \varphi$ to obtain $g_c(\vartheta, \varphi)$ as

$$g_c(\varphi, \vartheta) = g(\varphi, \vartheta, \omega) \Big|_{\omega=\pi-\varphi}. \quad (9)$$

By using the effect of rotation operator on the spherical harmonic coefficients of the signal as described in (5) and employing the orthonormal property of spherical harmonics, we can write the SO(3) convolution defined in (7) in spherical harmonic domain as

$$\begin{aligned} g(\varphi, \vartheta, \omega) &= \sum_{s=0}^L \sum_{t=-s}^s \sum_{t'=-s}^s (-1)^t (f)_s^{-t} (h)_s^{t'} D_s^{t,t'}(\varphi, \vartheta, \omega) \\ &= \sum_{s=0}^L \sum_{t=-s}^s \sum_{t'=-s}^s (-1)^t (f)_s^{-t} (h)_s^{t'} \\ &\quad \times e^{-it\varphi} d_s^{t,t'}(\vartheta) e^{-it'\omega}, \end{aligned} \quad (10)$$

where $L = \min(L_f, L_h)$. The direct computation of (10) involves three summation for sample points defined on three dimensional grid and therefore has the complexity $O(L^3 N^3)$. We note that the summations over t and t' involve complex exponentials and therefore can be computed efficiently by using FFT, that reduces the

overall complexity to $O(LN^3 \log_2 N)$, which is not better than the complexity of the direct quadrature case if $L \log_2 N > N$. However, it can be further lowered by using the factoring of rotation approach, originally presented in [15] and then applied for fast computation of SO(3) convolution [11].

By factoring the single rotation ϑ around y -axis as

$$\mathcal{D}(0, \vartheta, 0) = \mathcal{D}(-\pi/2, -\pi/2, \vartheta) \mathcal{D}(0, \pi/2, \pi/2) \quad (11)$$

and again incorporating the effect of rotation on spherical harmonic coefficients given in (5) and the definition of Wigner- D function in (6), we can write the Wigner- d in (10) as

$$d_s^{t,t'}(\vartheta) = i^{t-t'} \sum_{t''=-s}^s d_s^{t'',t}(\pi/2) d_s^{t'',t'}(\pi/2) e^{-it''\vartheta}, \quad (12)$$

which is used to express the SO(3) convolution formulated in (10) as

$$\begin{aligned} g(\varphi, \vartheta, \omega) &= \sum_{s=0}^L \sum_{t=-s}^s \sum_{t'=-s}^s \sum_{t''=-s}^s i^{3t-t'} (f)_s^{-t} (h)_s^{t'} \\ &\quad \times d_s^{t'',t}(\pi/2) d_s^{t'',t'}(\pi/2) e^{-it\varphi - it''\vartheta - it'\omega}, \end{aligned} \quad (13)$$

and the rearrangement of the terms yields

$$\begin{aligned} g(\varphi, \vartheta, \omega) &= \sum_{t=-s}^s \sum_{t'=-s}^s \sum_{t''=-s}^s e^{-it\varphi - it''\vartheta - it'\omega}, \\ &\quad \times \underbrace{\sum_{s=\max(|t|, |t'|, |t''|)}^L i^{3t-t'} (f)_s^{-t} (h)_s^{t'} d_s^{t'',t}(\pi/2) d_s^{t'',t'}(\pi/2)}_{I(t,t',t'')}. \end{aligned} \quad (14)$$

We note that the convolution output in (14) is a 3-dimensional FFT of $I(t, t', t'')$. The rotation ϑ , which accounts for the computation of Wigner- d function for all ϑ , can be performed as a rotation around z -axis by employing the factoring of rotation given in (11), the effect of which can be expressed using complex exponential and Wigner- d functions evaluated at $\pi/2$ only.

The evaluation of $I(t, t', t'')$ involves the summation over s for three dimensions t, t', t'' and therefore has the complexity $O(L^4)$. Using $I(t, t', t'')$, the SO(3) convolution output $g(\varphi, \vartheta, \omega)$ in (14) on the grid $\mathfrak{C}(N)$ can be computed in $O(N^3 \log_2 N)$. The overall complexity to evaluate SO(3) convolution is therefore

$O(L^4 + N^3 \log_2 N)$ which is better than the complexity of the exact quadrature case.

As we mentioned earlier, once the SO(3) convolution output $g(\varphi, \vartheta, \omega)$, it can be used to determine the commutative convolution output $g_c(\varphi, \vartheta)$ using (9). This method of using existing fast algorithms for SO(3) convolution to evaluate the commutative convolution enables the efficient computation in the harmonic space, however it involves redundancy in the computation as the current efficient method evaluates the SO(3) convolution output for all $\omega \in [0, 2\pi)$. Instead, we only need the SO(3) convolution output for $\omega = \pi - \varphi$. We remove this redundancy in the computation and propose fast algorithm in the next subsection.

Remark 1: Although both φ and ω are defined for $[0, 2\pi)$, we have deliberately chosen different sampling criterion along φ and ω during the definition of adopted SO(3) sampling scheme $\mathfrak{C}(N)$ (see Section III-A), where we have considered $2N + 1$ (odd number) samples along both φ and ω , but the sampling points along φ are symmetric around $\varphi = 0$ and the sampling points along ω are symmetric around $\omega = \pi$. This is in contrast to the conventional SO(3) sampling [17]. However, it is necessary here as we are evaluating the commutative convolution $g_c(\varphi, \vartheta)$ using SO(3) convolution $g(\varphi, \vartheta, \omega)$ with a constrain $\omega = \pi - \varphi$, which can only be applied if the proposed sampling $\mathfrak{C}(N)$ for SO(3) domain is used.

D. Proposed Fast Algorithm

Here, we propose a fast algorithm for the evaluation of commutative convolution on the sphere defined in (8). Following the harmonic domain formulation of SO(3) convolution in (13) and the relation between the convolution output on SO(3) and \mathbb{S}^2 , we can express $g_c(\vartheta, \varphi)$ as

$$g_c(\vartheta, \varphi) = \sum_{t''=-L}^L e^{-it''\vartheta} J(t'', \varphi) K(t'', \varphi), \quad (15)$$

where

$$J(t'', \varphi) = \sum_{t=-s}^s \sum_{s=\max(|t|, |t''|)}^L (-i)^t (f)_s^{-t} d_s^{t''t}(\pi/2) e^{-it\varphi} \quad (16)$$

and

$$K(t'', \varphi) = \sum_{t'=-s}^s \sum_{s=\max(|t'|, |t''|)}^L (i)^{t'} (h)_s^{t'} d_s^{t''t'}(\pi/2) e^{it'\varphi}. \quad (17)$$

We note that the constrain $\omega = \pi - \varphi$, which yields the commutativity also allows the decoupling of Wigner- D function in (15), so that the summation over t and t' , given in (16) and (17) respectively, can be computed independently. Since the computation of both $J(t'', \varphi)$ and $K(t'', \varphi)$ involves the summation over complex exponentials, we can employ FFT to compute the summations efficiently. For the convolution output on the grid $\mathfrak{S}(N)$ with $N \geq \max(L_f, L_h)$, both $J(t'', \varphi)$ and $K(t'', \varphi)$ for each t'' and for all $2N + 1$ points along φ can be computed in $O(LN \log_2 N)$ using FFT and the product of $J(t'', \varphi)$ and $K(t'', \varphi)$ can be computed in $O(N)$ computations for each t'' . Thus the overall complexity to obtain the product of $J(t'', \varphi)$ and $K(t'', \varphi)$ for each s and for each t'' is $O(LN \log_2 N)$ and for all t'' is $O(LN^2 \log_2 N)$. Finally, the sum over t'' can be computed efficiently in $O(N^2 \log_2 N)$ again using the FFT. Therefore, the overall complexity of proposed fast algorithm is $O(LN^2 \log_2 N)$, which is better than the complexities of both exact quadrature and semi-fast algorithm.

E. Computation of Wigner-d Function

We note that both semi-fast algorithm and fast algorithm require Wigner- d functions evaluated for argument $\pi/2$. By reviewing (14), (16) and (17), we note that we need to compute Wigner- d function $d_s^{t'',t}(\pi/2)$ on the entire (t'', t) plane. During implementation, the Wigner- d function $d_s^{t'',t}(\pi/2)$ can be computed on the plane (t'', t) for a given s by using the recursion method proposed in [18] with complexity $O(L^2)$, which do not alter the overall complexity of the algorithms (both semi-fast and fast).

IV. COMPUTATION TIME COMPARISON

In this section, we demonstrate and compare the computation time of semi-fast algorithm and fast algorithm to evaluate the commutative anisotropic convolution. We have implemented our algorithms using MATLAB, adopting the defined equiangular tessellations. We have recorded the computation time τ (in seconds) to evaluate convolution output $g_c(\vartheta, \varphi)$ in (8) on the grid $\mathfrak{S}(N)$ for $L = 32$ and $L = 64$ and for different values of N . We generate the band-limited test signal on the sphere by using uniformly distributed spherical harmonic coefficients with real and imaginary parts in the range of $[-1, 1]$. The numerical experiments on a 2.4 GHz Intel Xeon processor with 64 GB of RAM and the computation times are averaged over twenty test signals.

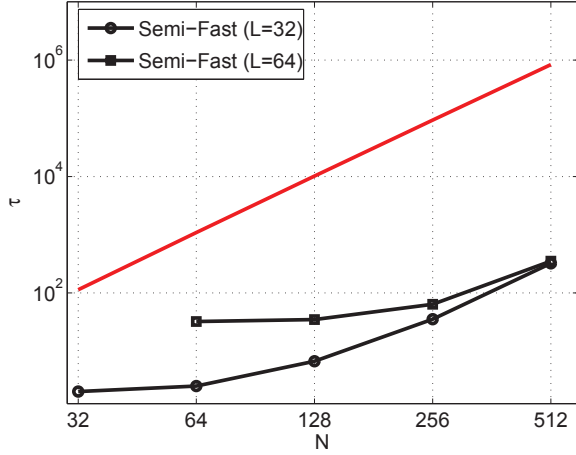


Fig. 1: The computation time τ in seconds for semi-fast algorithm to compute the convolution output $g_c(\vartheta, \varphi)$ in (8) on grid $\mathfrak{S}(N)$ for $L = 32$ and 64 . The computation time scales as $N^3 \log_2 N$ as indicated by red solid line (without markers).

The computation time τ taken by semi-fast algorithm to compute convolution output $g_c(\vartheta, \varphi)$ is shown in Fig. 1 on log-log axes for different values of N , where the computation time grows as $N^3 \log_2 N$. For the proposed fast algorithm, the computation time τ scales as $N^2 \log_2 N$ as shown in Fig. 2. For comparison, we have also plotted the computation time for both semi-fast and fast algorithm on a linear scale along time axis as shown in Fig. 3. We note that the simulation results agree with the theoretically evaluated computational complexities of the algorithms and thus corroborate the mathematical developments.

V. CONCLUSION

In this work, we have served an objective to efficiently evaluate the commutative anisotropic convolution on the 2-sphere. With the consideration of commutative anisotropic convolution as a special case of SO(3) convolution, we have first presented semi-fast algorithm based on the existing efficient techniques of SO(3) convolution. Later, we have proposed the fast algorithm which employs the factoring of rotation approach followed by the separation of variables technique. For the evaluation of the commutative convolution output on $O(N^2)$ samples on the 2-sphere, the proposed fast algorithm provides the saving of $O(N)$ in terms of the computational complexity over the semi-fast algorithm. Our proposed method evaluates the convolution of the signals given in spectral domain, which makes the evaluation exact

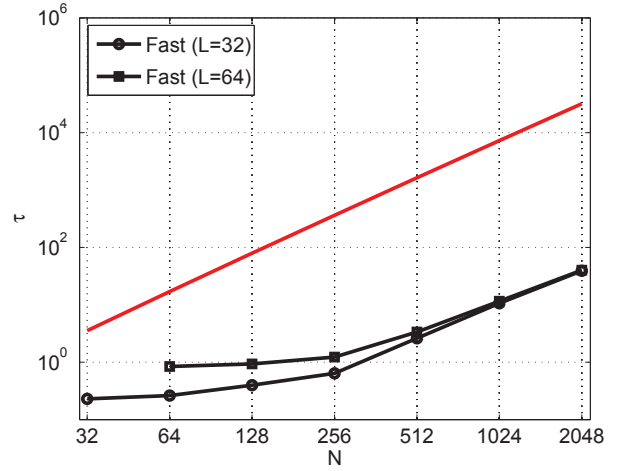


Fig. 2: The computation time τ in seconds for proposed fast algorithm to compute the convolution output $g_c(\vartheta, \varphi)$ in (8) on grid $\mathfrak{S}(N)$ for $L = 32$ and 64 . The computation time scales as $N^2 \log_2 N$ as indicated by red solid line (without markers).

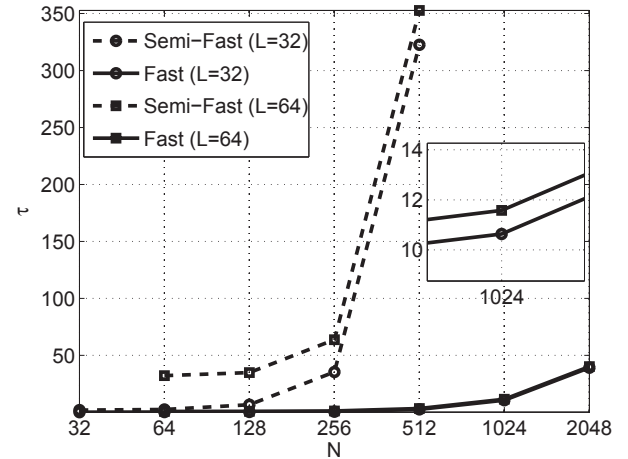


Fig. 3: The comparison of the computation time τ (in seconds) for semi-fast and fast algorithm to compute the convolution output $g_c(\vartheta, \varphi)$ in (8) on grid $\mathfrak{S}(N)$ for $L = 32$ and 64 .

with the only assumption that one of the two signals involved in the convolution is band-limited. We have also presented simulation results to verify the theoretical improvement in the computational complexity.

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