Spatial Correlation from Multipath with 3D Power Distributions having Rotational Symmetry

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Abstract—In this paper we give a general expression for the 3D spatial correlation experienced between two sensors in 3D-space for the class of normalized power distributions (representing farfield multipath sources) having a rotational symmetry about their mean direction axis. A general expansion for the 3D spatial correlation is presented and interpreted in terms of an associated eigenfunction equation. This enables us to develop closed-form coefficient expressions for the spatial correlation for a number of distributions such as the Gauss-Weierstrass kernel based distribution and the previous known results for the von Mises-Fisher power distribution. Analytical results generated fully account for the effect of varying in the relative orientation between the sensors in 3D and the power distribution mean direction which can be arbitrarily oriented. The results provide information on placement of sensors to reduce correlation effects.

Index Terms—wireless channel, spatial correlation, multipath, von Mises distribution, von Mises-Fisher distribution, Gauss-Weierstrass distribution, spherical harmonics.

I. INTRODUCTION

General expressions for the spatial correlation between two sensors for arbitrary multipath power distributions were given in [1] for both the 2D case which restricts sensors and multipath to the horizontal plane and the unrestricted 3D case. The 3D case permits multipath to arrive with different elevations and azimuths, and the sensors to be at different heights and orientations. In the 2D case, closed-form expressions were given for the coefficients of number of multipath distributions: uniform, uniform azimuth limited [2], power of cosine [3], von Mises [4], truncated gaussian [5], truncated Laplacian [6], etc., in a spatial correlation expansion which converges rapidly. These 2D multipath distributions have the property that they are symmetric about their mean direction of arrival. For other distributions one can resort to numerical integration methods to compute the spatial correlation but there is a clear preference for analytical results where the effects of parameters is revealed.

For the 3D case, which is our focus in this paper also, a general expression was given in [1] which identified the spherical harmonic coefficients of the multipath power distribution with the coefficients used in an expansion for the spatial

correlation. Then to obtain a closed-form coefficient spatial correlation expansion one only needs to find expressions for the power distribution which admit suitable expansions in spherical harmonics or Legendre polynomials. In [1] two examples were given of such a procedure, the first was the well-known omnidirectional case which leads to the famous sinc spatial correlation

$$\rho(z_1 - z_2) = \frac{\sin(k|z_1 - z_2|)}{k|z_1 - z_2|}$$

where $k = 2\pi/\lambda$, λ is the wavelength and $|z_1 - z_2|$ is the euclidean distance between spatial points z_1 and z_2. Because of the zero crossings, this formula can be seen as a theoretical basis for the half wavelength, $\lambda/2$, sensor spacing rule-of-thumb. The other example given in [1] was for multipath that was uniformly distributed over a limited range of elevations, which is somewhat contrived and unlikely to have practical significance. More recently, it was shown that the 3D von Mises-Fisher distribution admits a suitable closed-form expansion in spherical harmonics in the form of a recursion [7] and in closed-form using half-integer-order modified Bessel functions of the first kind in [8]. This latter case is a direct 3D analog of the 2D von Mises distribution case developed in [1], which uses integer-order modified Bessel functions of the first kind.

The von Mises-Fisher distribution has rotational symmetry about its mean direction which can have both azimuth and elevation components. This rotational symmetry is the key attribute that permits simplification and we exploit this to develop results for distributions more general than the von Mises-Fisher distribution.

Similarly we make no restriction on the placement of the sensors in 3D space. It is this type of distribution which we target so as to develop analytical results where the effect of various parameters can be revealed.

In this paper, we develop a general expression for the spatial correlation for 3D distributions having rotational symmetry about some axis including as a special case the von Mises-Fisher distribution. Firstly, the general expression reveals features and therefore insights in common to all cases. Then for a number of special cases, reflecting distributions more realistic in nature, we provide closed-form expressions for the

spatial correlation from which it is possible to characterize convergence.

II. ROTATIONALLY SYMMETRIC DISTRIBUTIONS

A distribution $f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}})$ on the 2-sphere, denoted $\mathbb{S}^2$, with random unit vector $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$ with rotational symmetry about some axis $\widehat{\boldsymbol{\mu}} \in \mathbb{S}^2$, see Fig. 1, can be written in terms of a real-valued univariate function

$$f(z), \quad \text{defined on } z \in [-1, +1]$$

as follows

$$f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}}) = f(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\mu}}). \tag{1}$$

If $\widehat{\boldsymbol{\eta}} \in \mathbb{S}^2$ denotes the north pole, then a symmetric function $f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\eta}}) = f(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\eta}})$ is called azimuthally symmetric.

The requirements to be a valid distribution (with equivalent conditions on the univariate function) are: and non-negativity:

$$f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}}) \geq 0, \ \forall \widehat{\boldsymbol{x}} \in \mathbb{S}^2 \iff f(z) \geq 0, \ \forall z \in [-1, +1]. \tag{2}$$

and normalization

$$\int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}}) \, ds(\widehat{\boldsymbol{x}}) = 1 \iff 2\pi \int_{-1}^{+1} f(z) \, dz = 1 \tag{3}$$

where $ds(\widehat{\boldsymbol{x}}) = \sin\theta \, d\theta \, d\varphi$ is the uniform surface measure on the 2-sphere (see Appendix A). The equivalence in (3) can be proven with change of variables and the substitution $z = \widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\mu}}$.

Example: The von Mises-Fisher distribution on 2-sphere can be defined in terms of the univariate function

$$f_\kappa(z) \triangleq \frac{\kappa \exp(\kappa z)}{4\pi \sinh \kappa}, \quad z \in [-1, +1], \tag{4}$$

leading to

$$\begin{aligned} f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}}, \kappa) &= f_\kappa(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\mu}}) \\ &\triangleq \frac{\kappa \exp(\kappa \, \widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\mu}})}{4\pi \sinh \kappa} \end{aligned} \tag{5}$$

where $\widehat{\boldsymbol{\mu}} \in \mathbb{S}^2$ is the mean direction, $\kappa \geq 0$ is the concentration parameter, $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$ is a random unit vector on the 2-sphere, and the remaining terms serve to normalize the distribution.

III. SPHERICAL HARMONIC EXPANSION

Appendix A gives the definition of spherical harmonics, inner product, and related concepts used below. The objective will be to provide the spherical harmonic coefficients and spherical harmonic expansion of a rotationally symmetric function on the 2-sphere with axis of symmetry $\widehat{\boldsymbol{\mu}} \in \mathbb{S}^2$, and this task is simplified by the addition theorem of the spherical harmonics [9], [10]

$$\sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\boldsymbol{x}}) \overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})} = \frac{(2\ell+1)}{4\pi} P_\ell(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\mu}}). \tag{6}$$

So we expand the univariate function $f(\cdot)$ in terms of Legendre polynomials with adjustments to directly exploit (6) and simplify later development:

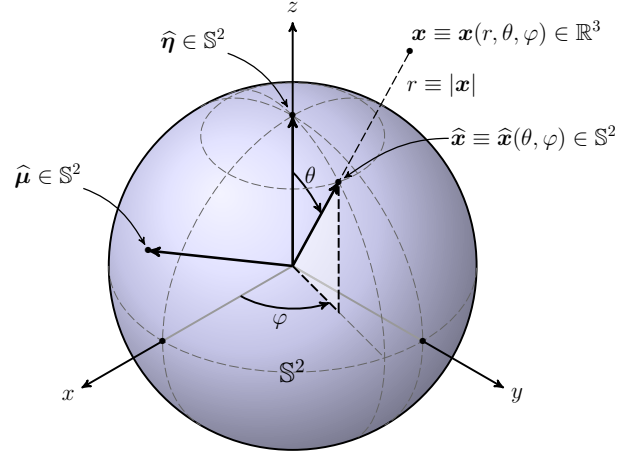$$f(z) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \lambda_\ell P_\ell(z), \tag{7}$$

Fig. 1: Coordinate system showing a point in 3D space $\boldsymbol{x}$, a unit vector $\widehat{\boldsymbol{x}}$, which lies on the 2-sphere, $\mathbb{S}^2$, and the spherical polar coordinate system with $\theta \in [0, \pi]$ the co-latitude, $\varphi \in [0, 2\pi)$ the longitude and $r \equiv |\boldsymbol{x}|$ the radius.

where the eigenvalues are (the terminology will be justified later)

$$\lambda_\ell = 2\pi \int_{-1}^{+1} f(z) P_\ell(z) \, dz. \tag{8}$$

Since $f(z)$ is real-valued then λ_ℓ is also real-valued but can positive, negative or zero. With $\ell = 0$ in (8) then we recover (3). This means the normalization is equivalent to $\lambda_0 = 1$.

So combining (6), (8) into (7) we can infer the spherical harmonic expansion of any symmetric distribution

$$f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}}) = f(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\mu}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \lambda_\ell \, \overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})} \, Y_\ell^m(\widehat{\boldsymbol{x}}),$$

which is of the general form of a spherical harmonic expansion

$$f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f_{\widehat{\boldsymbol{\mu}}})_\ell^m \, Y_\ell^m(\widehat{\boldsymbol{x}}), \tag{9}$$

where the spherical harmonic coefficients are

$$(f_{\widehat{\boldsymbol{\mu}}})_\ell^m \triangleq \langle f(\cdot \, ; \widehat{\boldsymbol{\mu}}), Y_\ell^m \rangle = \lambda_\ell \, \overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})}. \tag{10}$$

So in summary, given a distribution $f(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{\mu}})$ with axis of symmetry in some direction $\widehat{\boldsymbol{\mu}} \in \mathbb{S}^2$ then the formula to determine the spherical harmonic coefficients is (10) which requires us to compute the eigenvalues (8) from the associated univariate function $f(z)$.

A. Properties and Examples

Example 1: von Mises-Fisher Distribution: The eigenvalues for the von Mises-Fisher distribution are

$$\lambda_\ell(\kappa) = 2\pi \int_{-1}^{+1} f_\kappa(z) P_\ell(z) \, dz. \tag{11}$$

where the univariate function is given in (4). This expression can be computed recursively as shown in [7]

$$\lambda_{\ell+1}(\kappa) = \lambda_{\ell-1}(\kappa) - \frac{(2\ell+1)}{\kappa} \lambda_\ell(\kappa), \quad \ell = 1, 2, \dots.$$

with initial values $\lambda_0(\kappa) = 1$ and $\lambda_1 = (\kappa) = \coth\kappa - 1/\kappa$, leading to

$$\left\{\lambda_\ell(\kappa)\right\}_{\ell=0}^\infty = \left\{1, \coth\kappa - \frac{1}{\kappa}, \frac{\kappa^2 - 3\kappa\coth\kappa + 3}{\kappa^2}, \ \ldots\right\}.$$

Further, it can be established that $\lambda_\ell(\kappa) > 0$, for all $\ell = 0, 1, 2, \ldots$, and for all $\kappa \geq 0$ [10, p.394]. In [8], [11], [12] it was shown that these coefficients can also be expressed as

$$\lambda_\ell(\kappa) = \frac{I_{\ell+1/2}(\kappa)}{I_{1/2}(\kappa)} = \sqrt{\frac{\pi\kappa}{2}}\frac{I_{\ell+1/2}(\kappa)}{\sinh\kappa}$$

where $I_{\ell+1/2}(\cdot)$ is a half-integer-order modified Bessel function of the first kind.

With the above, the spherical harmonic coefficients are

$$(f_{\widehat{\boldsymbol{\mu}},\kappa})_\ell^m \triangleq \langle f(\cdot\,;\widehat{\boldsymbol{\mu}},\kappa), Y_\ell^m\rangle = \frac{I_{\ell+1/2}(\kappa)}{I_{1/2}(\kappa)}\overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})}, \quad (12)$$

and the spherical harmonic expansion is

$$\frac{\kappa\exp(\kappa\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{\mu}})}{4\pi\sinh\kappa} = \sum_{\ell=0}^\infty\sum_{m=-\ell}^\ell \frac{I_{\ell+1/2}(\kappa)}{I_{1/2}(\kappa)}\overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})}\,Y_\ell^m(\widehat{\boldsymbol{x}}).$$

Example 2: Azimuthally Symmetric Distribution: Aligning the axis of symmetry with the z-axis or north pole means setting $\widehat{\boldsymbol{\mu}} = \widehat{\boldsymbol{\eta}} \triangleq [0,0,1]'$. Spherical harmonics for $m = 0$ also have rotational symmetry about the z-axis and these are the only terms that are required in the spherical harmonic expansion as we now show.

We use the simple identity (using the definitions given in Appendix A):

$$Y_\ell^m(\widehat{\boldsymbol{\eta}}) = \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m,0}.$$

Then

$$(f_{\widehat{\boldsymbol{\eta}}})_\ell^m \triangleq \langle f(\cdot\,;\widehat{\boldsymbol{\eta}}), Y_\ell^m\rangle = \lambda_\ell\overline{Y_\ell^m(\widehat{\boldsymbol{\eta}})} = \lambda_\ell\sqrt{\frac{2\ell+1}{4\pi}}\delta_{m,0}$$

and so the spherical harmonic expansion is

$$\begin{aligned}
f(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{\eta}}) = f(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{\eta}}) &= \sum_{\ell=0}^\infty\sum_{m=-\ell}^\ell (f_{\widehat{\boldsymbol{\eta}}})_\ell^m\,Y_\ell^m(\widehat{\boldsymbol{x}}) \\
&= \sum_{\ell=0}^\infty\sum_{m=-\ell}^\ell \lambda_\ell\sqrt{\frac{2\ell+1}{4\pi}}\delta_{m,0}\,Y_\ell^m(\widehat{\boldsymbol{x}}) \\
&= \sum_{\ell=0}^\infty\sqrt{\frac{2\ell+1}{4\pi}}\lambda_\ell\,Y_\ell^0(\widehat{\boldsymbol{x}})
\end{aligned}$$

which can also be expressed in terms of Legendre polynomials equivalent to (7) with $z = \widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{\eta}}$.

Example 3: Gauss-Weierstrass Distribution: A special case of this azimuthally symmetric distribution is the Gauss-Weierstrass kernel [13] with concentration parameter $\kappa \geq 0$

$$f(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{\eta}},\kappa) = \sum_{\ell=0}^\infty\sqrt{\frac{2\ell+1}{4\pi}}e^{-\ell(\ell+1)/2\kappa}\,Y_\ell^0(\widehat{\boldsymbol{x}})$$

where we see $\lambda_\ell(\kappa) \equiv e^{-\ell(\ell+1)/2\kappa}$ and this is approximates the von Mises-Fisher distribution with mean direction $\widehat{\boldsymbol{\eta}}$ in the

limit $\kappa \gg 1$, see [7]. If we use a general mean direction $\widehat{\boldsymbol{\mu}}$, instead of $\widehat{\boldsymbol{\eta}}$, the rotated Gauss-Weierstrass kernel becomes

$$f(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{\mu}},\kappa) = \sum_{\ell=0}^\infty\sum_{m=-\ell}^\ell e^{-\ell(\ell+1)/2\kappa}\overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})}\,Y_\ell^m(\widehat{\boldsymbol{x}}).$$

The Gauss-Weierstrass kernel spatial expression does not simplify to a known closed-form (spatial) expression. However, it does have a simple spectral or eigenvalue characterization. Spatially they are unimodal with adjustable width, and when they are narrower they get asymptotically closer to the von Mises-Fisher distribution [7].

B. Integral Equation, Eigenvalues and Eigenfunctions

The underlying eigen-structure of the problem under study explains why rotationally symmetric functions lead to an elegant solution and also explain why it is unlikely the methods can be easily extended to more general asymmetric functions.

Theorem 1: Define the integral operator $\mathcal{F}$ as follows

$$\left(\mathcal{F}h\right)(\widehat{\boldsymbol{x}}) = \int_{\mathbb{S}^2} F(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}})h(\widehat{\boldsymbol{y}})\,ds(\widehat{\boldsymbol{y}})$$

with kernel

$$F(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}}) \triangleq f(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{y}}) = \sum_{\ell=0}^\infty\sum_{m=-\ell}^\ell \lambda_\ell Y_\ell^m(\widehat{\boldsymbol{x}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}. \quad (13)$$

Then

$$\left(\mathcal{F}Y_\ell^m\right)(\widehat{\boldsymbol{x}}) = \lambda_\ell Y_\ell^m(\widehat{\boldsymbol{x}}) \quad (14)$$

which reveals the λ_ℓ, (8), are the eigenvalues corresponding to eigenfunctions $Y_\ell^m(\widehat{\boldsymbol{x}})$ of integral operator $\mathcal{F}$.

Further, with the finite energy condition

$$\int_{-1}^{+1} f^2(z)\,dz < \infty, \quad (15)$$

the operator $\mathcal{F}$ is compact and self adjoint.

Proof: From (10), $\langle f(\cdot\,;\widehat{\boldsymbol{\mu}}), Y_\ell^m\rangle = \lambda_\ell\overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})}$, is the same as

$$\int_{\mathbb{S}^2} f(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{\mu}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\,ds(\widehat{\boldsymbol{y}}) = \lambda_\ell\overline{Y_\ell^m(\widehat{\boldsymbol{\mu}})}.$$

Then from the definitions given in Appendix A

$$\overline{Y_\ell^m(\widehat{\boldsymbol{x}})} = (-1)^m Y_\ell^{-m}(\widehat{\boldsymbol{x}}),$$

and from symmetry $f(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{\mu}}) = f(\widehat{\boldsymbol{\mu}};\widehat{\boldsymbol{y}})$ we get (14).

Self-adjointness of operator $\mathcal{F}$ follows from Hermitian symmetry, due to $f(\cdot)$ being real-valued,

$$f(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{\mu}}) = \overline{f(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{\mu}})} = \overline{f(\widehat{\boldsymbol{\mu}};\widehat{\boldsymbol{y}})} \implies \mathcal{F} = \mathcal{F}^*.$$

Then from [10, p.357], the finite energy condition (15) can be shown to be equivalent to finite energy (sum square) of the eigenvalues (summed with multiplicity) by the Parseval relation for Legendre polynomials. This implies the kernel (13) is Hilbert-Schmidt and, therefore, operator $\mathcal{F}$ is compact. $\square$

This justifies seeking a spherical harmonic representation for rotationally symmetric distributions $f(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{\mu}})$ in the first place, Section III, because they are eigenfunctions. Similarly, making the coefficients identical to the eigenvalues makes the development cogent.

IV. Spatial Correlation

A. Rotationally Symmetric Distribution Case

We review the development in [1]. The spatial correlation between two narrowband (fixed wavenumber k) complex-valued signals $s_1(t), s_2(t)$ at points $z_1, z_2 \in \mathbb{R}^3$ is given by

$$\rho(z_1, z_2) = \frac{\mathbb{E}\{s_1(t)\,\overline{s_2(t)}\}}{\mathbb{E}\{s_1(t)\,\overline{s_1(t)}\}} \qquad (16)$$

where $\mathbb{E}\{\cdot\}$ denotes expectation over the random complex gains from the multipath scattering [1]. Then with a power distribution $f(\widehat{x}; \widehat{\mu})$, normalized according to (3), representing the average power density as a function of direction $\widehat{x} \in \mathbb{S}^2$ in the farfield of the two points $z_1, z_2 \in \mathbb{R}^3$ then the spatial correlation, (16), becomes spatially wide-sense stationary in the form

$$\rho(z_1 - z_2; \widehat{\mu}) = \int_{\mathbb{S}^2} f(\widehat{x}; \widehat{\mu}) e^{ik(z_1 - z_2)\cdot\widehat{x}}\, ds(\widehat{x}). \qquad (17)$$

The wide-sense stationarity does not depend on the rotational symmetry of $f(\widehat{y}; \widehat{\mu})$, and the above result is quite general.

Then by employing a spherical harmonic expansion of plane waves of the continuous superposition in (17) one arrives at

$$\rho(z_1 - z_2; \widehat{\mu}) = 4\pi \sum_{\ell=0}^{\infty} i^\ell j_\ell\big(k|z_1 - z_2|\big) \times$$
$$\sum_{m=-\ell}^{\ell} (f_{\widehat{\mu}})_\ell^m Y_\ell^m\left(\frac{z_1 - z_2}{|z_1 - z_2|}\right), \quad (18)$$

where $(f_{\widehat{\mu}})_\ell^m$ are the spherical harmonic coefficients of the distribution $f(\cdot\,; \widehat{\mu})$, given in (10).

B. Spatial Correlation Expansion

We specify the results in [1] for 3D scattering exhibiting rotational symmetry about some axis of symmetry $\widehat{\mu} \in \mathbb{S}^2$ expressed in the context of the eigen-problem. We gather our main theoretical findings into a self-contained statement.

Theorem 2: Let the multipath be defined by the normalized power distribution with rotational symmetry

$$f(\widehat{x}; \widehat{\mu}) = f(\widehat{x} \cdot \widehat{\mu})$$

where $\widehat{x} \in \mathbb{S}^2$ is the direction, $\widehat{\mu} \in \mathbb{S}^2$ is the mean direction, and $f(\cdot)$ is a non-negative, real univariate function with domain $[-1, +1]$. Then the spatial correlation between points $z_1, z_2 \in \mathbb{R}^3$ depends only on $z \triangleq z_1 - z_2$, is spatially wide-sense stationary, and is given by

$$\rho(z; \widehat{\mu}) = j_0\big(k|z|\big) + \sum_{\ell=1}^{\infty}(2\ell + 1)\, i^\ell \times$$
$$\lambda_\ell\, P_\ell(\widehat{z} \cdot \widehat{\mu})\, j_\ell\big(k|z|\big), \quad (19)$$

where $z \in \mathbb{R}^3$, $\widehat{z} = z/|z| \in \mathbb{S}^2$, $|z|$ is the Euclidean distance,

$$\lambda_\ell = 2\pi \int_{-1}^{+1} f(z) P_\ell(z)\, dz, \quad \ell = 0, 1, \ldots, \qquad (8)$$

and

$$f(z) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty}(2\ell + 1)\lambda_\ell P_\ell(z), \qquad (7)$$

noting that $\lambda_0 = 1$.

Proof: Using (10) and the addition theorem (6), (18) becomes

$$\rho(z; \widehat{\mu}) = 4\pi \sum_{\ell=0}^{\infty} i^\ell j_\ell\big(k|z|\big) \sum_{m=-\ell}^{\ell} \lambda_\ell\, \overline{Y_\ell^m(\widehat{\mu})}\, Y_\ell^m(\widehat{z}),$$
$$= \sum_{\ell=0}^{\infty}(2\ell + 1)\, i^\ell \lambda_\ell\, P_\ell(\widehat{z} \cdot \widehat{\mu})\, j_\ell\big(k|z|\big).$$

which can be written as in (19) which separates out the $\ell = 0$ term. $\qquad \square$

Comments

1) Note that the $\ell = 0$ term is a spherical Bessel function equivalent to a sinc function:

$$j_0\big(k|z|\big) = \frac{\sin\big(k|z|\big)}{k|z|} = \lambda\,\frac{\sin\big(2\pi|z|/\lambda\big)}{2\pi|z|}.$$

This term, common to all spatial correlation expansions (because the power distribution is nonnegative there is always a positive DC term which when normalized is unity), is the 3D omnidirectional contribution to spatial correlation, corresponding to $\lambda_0 = 1$, and has zero crossings at $\lambda/2$ spacings except at zero where the correlation is unity as shown in Fig. 2. This is equivalent to the von Mises-Fisher distribution case with $\kappa = 0$.

2) The relative orientation of the vector joining the two sensors and the mean direction (axis of rotational symmetry) of the power distribution is clearly delineated in the expression (19) with the appearance of the term $\widehat{z} \cdot \widehat{\mu}$. This is the only way that $\widehat{\mu}$ enters the formula.

3) The shape of the power distribution, (7), is defined by the λ_ℓ which are given by (8).

C. von Mises-Fisher Distribution

For $f(\widehat{x}; \widehat{\mu}, \kappa)$ in (5) the univariate functions is given by $f_{\widehat{\mu}}(z)$ in (4) which has eigenvalues $\lambda_\ell(\kappa)$ given by (11). Then the spatial correlation (19) is given by

$$\rho(z; \widehat{\mu}, \kappa) = \sum_{\ell=0}^{\infty}(2\ell+1)\, i^\ell\, \frac{I_{\ell+1/2}(\kappa)}{I_{1/2}(\kappa)}\, P_\ell(\widehat{z}\cdot\widehat{\mu})\, j_\ell\big(k|z|\big). \quad (20)$$

This expression can be compared with the structurally similar

$$f(\widehat{x}; \widehat{\mu}, \kappa) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty}(2\ell + 1)\frac{I_{\ell+1/2}(\kappa)}{I_{1/2}(\kappa)} P_\ell(\widehat{x} \cdot \widehat{\mu}), \qquad (21)$$

which follows from (1), (7). This is plotted in Fig. 3, in the form $f_\kappa(\cos\theta)$, for a range of κ.

Corresponding to the highlighted cases in Fig. 3 we have the spatial correlation (20) plotted in Fig. 4, for $\kappa = 1, 2, 4, 8$ and Fig. 2 for $\kappa = 0$. The total number of terms in the expansion required to yield the accuracy shown in the plots was $L = 1, 6, 7, 10, 13, 18$, meaning the sum is over $\ell = 0, 1, \ldots, L-1$,
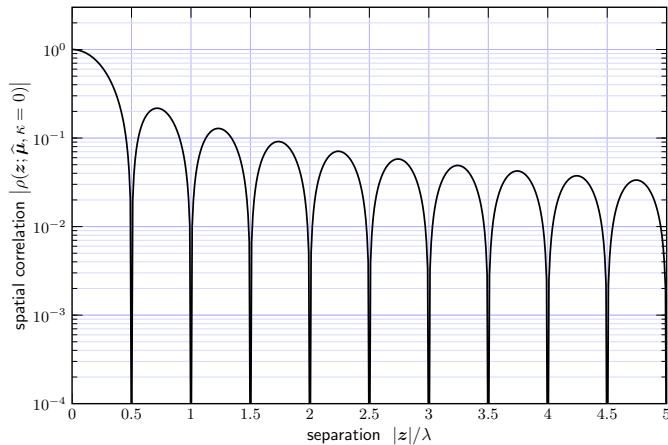
Fig. 2: Magnitude of the spatial correlation $\left|\rho(\boldsymbol{z};\widehat{\boldsymbol{\mu}},\kappa=0)\right|$ for the von Mises-Fisher distribution with concentration parameter $\kappa=0$, which is the 3D omnidirectional case. This correlation is independent of $\widehat{\boldsymbol{z}}\cdot\widehat{\boldsymbol{\mu}}$ and where $\boldsymbol{z}$ is the vector between the two sensors and $\widehat{\boldsymbol{\mu}}$ is the mean direction.
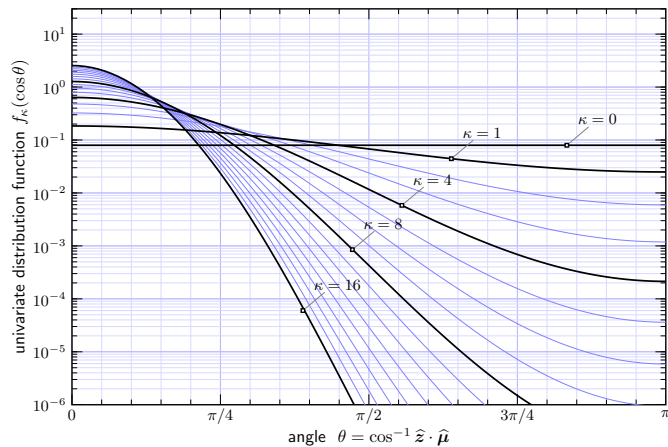

Fig. 3: Von Mises-Fisher univariate distribution functions, $f_\kappa(\cos\theta)$ for $\kappa=0,1,4,8,16$ (intermediate values increment by 1) plotted against $\theta=\cos^{-1}\widehat{\boldsymbol{z}}\cdot\widehat{\boldsymbol{\mu}}$.

for $\kappa=0,1,2,4,8,16$, respectively. The plots show the effect due to the effective width of the distribution around the mean direction. Naturally as κ gets smaller the width gets larger and vice versa. Also by varying the $\widehat{\boldsymbol{z}}\cdot\widehat{\boldsymbol{\mu}}$ we see the full effect of varying in the relative orientation between the sensors $\boldsymbol{z}=\boldsymbol{z}_1-\boldsymbol{z}_2$ and the distribution mean direction $\widehat{\boldsymbol{\mu}}$. These results are fully analytical and not simulated (which would be very time consuming to generate accurate data). They are quantitative and fully consistent with qualitative expectations.

Note that the term $j_\ell(k|\boldsymbol{z}|)$, which enters the spatial correlation expression (20) but is absent in the distribution expansion (21), serves to speed convergence in the following sense which relies on a result developed for the Bessel functions in [14], [15]. For the spherical Bessel functions the result says that for

$$\ell\geq\lceil\pi e|z|/\lambda\rceil$$

the $j_\ell(k|\boldsymbol{z}|)$ contribution is very small effectively truncating the series [16]. However, the remainder of the series coefficient also becomes smaller as $\ell\to\infty$ and an effective truncation can have few terms than that determined by $j_\ell(k|\boldsymbol{z}|)$ alone.

D. Lebedev Distribution

Non-negative real-valued functions on $[-1,+1]$ which have a simple Legendre polynomial expansion lead to closed-form 3D distributions and thereby closed-form spatial correlation expansions. An example is modifying a function given in [17], and [18, (8.922.6)], and developed in [19]

$$f(\boldsymbol{z};\widehat{\boldsymbol{\mu}},\eta)=\left(\frac{1}{4\pi}+\frac{\eta}{12\pi}\right)-\frac{\eta}{8\pi}\sqrt{\frac{1-\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{\mu}}}{2}}$$
$$=\frac{1}{4\pi}+\eta\sum_{\ell=1}^{\infty}\frac{P_\ell(\widehat{\boldsymbol{z}}\cdot\widehat{\boldsymbol{\mu}})}{(2\ell-1)(2\ell-3)},\quad\eta\in[0,6],$$

which, therefore, has a spatial correlation expansion

$$\rho(\boldsymbol{z};\widehat{\boldsymbol{\mu}},\eta)=j_0(k|\boldsymbol{z}|)+\eta\sum_{\ell=1}^{\infty}\frac{i^\ell P_\ell(\widehat{\boldsymbol{z}}\cdot\widehat{\boldsymbol{\mu}})}{(2\ell-1)(2\ell-3)}\,j_\ell(k|\boldsymbol{z}|)$$

where $\eta\in[0,6]$ ensures the distribution is non-negative.

Further closed-form spatial correlation expansions can be inferred from the results in [19] as we discuss in the conclusions.

V. CONCLUSIONS

Prior to this paper the von Mises distribution had been presented as a versatile correlation function in the 2D context [4] along with other distributions that also had a mirror symmetry property [5], [6]. In the 3D case, closed-form spatial correlation function having a power distribution with rotational symmetry property were limited to two simple cases considered in [1] and the von Mises-Fisher distribution in [8]. This paper has given a general expansion for the 3D spatial correlation for the class of normalized power distributions representing farfield multipath sources having a rotational symmetry about their mean direction. The main theoretical result, Theorem 2, also gives good insight into the interplay between the positioning of the sensors and the distribution mean direction. In Theorem 1, we have fully accounted for the spherical harmonic eigenfunction and eigenvalue structure of the spatial correlation problem for power distributions with rotational symmetry property.

Finding other suitable univariate functions with closed-form coefficient expansions has an interesting relationship with finding closed-form reproducing kernels on the 2-sphere [19], [20]. Functions which work as the basis for constructing closed-form reproducing kernels also mostly work as univariate functions for the purpose of defining closed-form multipath power distributions. We gave a few examples in this paper but others given in [19] also work as alternatives. In the current problem it is the spatial function that needs to be non-negative as it represents a power/probability distribution but the spectral description expressed through the eigenvalues can be positive or negative. In contrast, for the reproducing
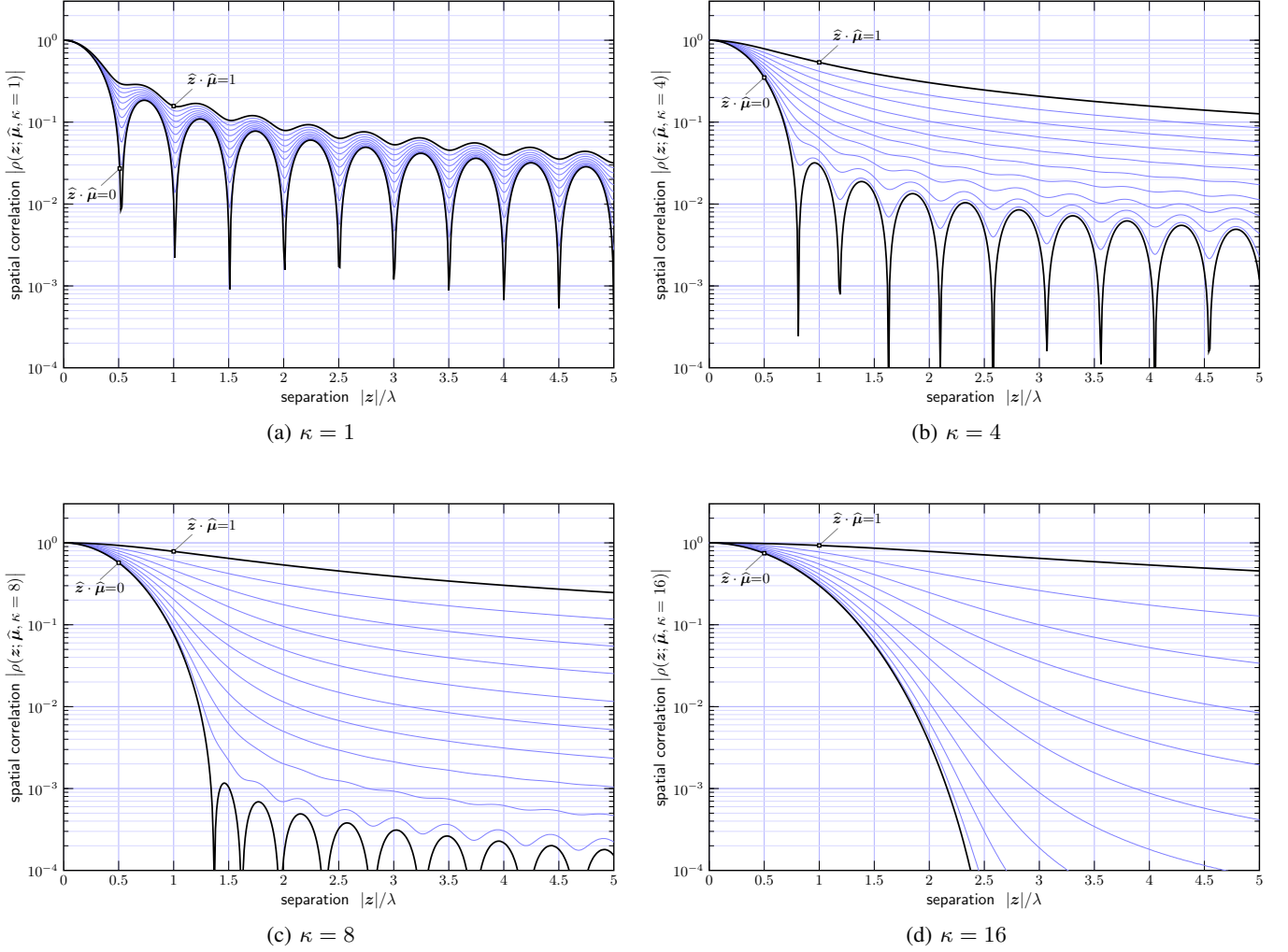
(a) $\kappa = 1$

(b) $\kappa = 4$

(c) $\kappa = 8$

(d) $\kappa = 16$

Fig. 4: Magnitude of the spatial correlation $\left|\rho(\boldsymbol{z}; \widehat{\boldsymbol{\mu}}, \kappa)\right|$ for the von Mises-Fisher distribution with concentration parameter κ and for a range of values of $\widehat{\boldsymbol{z}} \cdot \widehat{\boldsymbol{\mu}} \in \{0, 0.1, 0.2, \ldots, 1.0\}$ and where $\boldsymbol{z}$ is the vector between the two sensors and $\widehat{\boldsymbol{\mu}}$ is the mean direction. The labelled curves are broadside: $\widehat{\boldsymbol{z}} \cdot \widehat{\boldsymbol{\mu}} = 0$ and end-fire: $\widehat{\boldsymbol{z}} \cdot \widehat{\boldsymbol{\mu}} = 1$.

kernel case the eigenvalues need to be positive and the spatial function, being the kernel itself, need not be a positive function but predominantly so. The effectiveness of synthesizing non-unimodal power distributions [8] from a reproducing kernel perspective is yet to be determined.

APPENDIX A
SPHERICAL HARMONICS

Define the 2-sphere by $\mathbb{S}^2 \triangleq \{\boldsymbol{x} \in \mathbb{R}^3 \colon |\boldsymbol{x}| = 1\}$ and the inner product of two functions whose domain is the 2-sphere

$$\langle f, g \rangle \triangleq \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}}) \overline{g(\widehat{\boldsymbol{x}})} \, ds(\widehat{\boldsymbol{x}}), \tag{22}$$

where $\widehat{\boldsymbol{x}} \triangleq (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)' \in \mathbb{S}^2 \subset \mathbb{R}^3$ and $ds(\widehat{\boldsymbol{x}}) = \sin\theta \, d\theta \, d\varphi$ is the uniform surface measure satisfying $\int_{\mathbb{S}^2} ds(\widehat{\boldsymbol{x}}) = 4\pi$. Finite energy functions are those that satisfy the bounded induced norm condition

$$f \in L^2(\mathbb{S}^2) \iff \|f\| \triangleq \langle f, f \rangle^{1/2} < \infty.$$

In this work we require the spherical harmonics. They are defined through

$$Y_\ell^m(\theta, \varphi) \triangleq \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi} \equiv Y_\ell^m(\widehat{\boldsymbol{x}}),$$

where $\ell \in \{0, 1, \ldots\}$ is the degree, $m \in \{-\ell, -\ell+1, \ldots, \ell\}$ is the order, the associated Legendre functions are

$$P_\ell^m(z) \triangleq \frac{(-1)^m}{2^\ell \ell!} (1-z^2)^{m/2} \frac{d^{\ell+m}}{dz^{\ell+m}} (z^2-1)^\ell,$$
$$m \in \{0, 1, \ldots, \ell\}$$

and satisfy

$$P_\ell^{-m}(z) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(z), \quad m \in \{0, 1, \ldots, \ell\},$$

which enables the determination of the spherical harmonics for $m < 0$. For $m = 0$ the associated Legendre functions reduce to the standard Legendre polynomials which are denoted $P_\ell(z)$.

For each $f \in L^2(\mathbb{S}^2)$ we have expansion in the spherical harmonics

$$f(\widehat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_\ell^m \, Y_\ell^m(\widehat{\boldsymbol{x}}),$$

where the spherical harmonic coefficients are

$$(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle.$$

The equality in the expansion is understood in the sense of convergence in the mean [10].

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