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Commutative Anisotropic Convolution on the 2-Sphere

Parastoo Sadeghi, Rodney A. Kennedy, and Zubair Khalid

***Abstract*—We develop a new type of convolution between two signals on the 2-sphere. This is the first type of convolution on the 2-sphere which is commutative. Two other advantages, in comparison with existing definitions in the literature, are that 1) the new convolution admits anisotropic filters and signals and 2) the domain of the output remains on the sphere. Therefore, the new convolution well emulates the conventional Euclidean convolution. In addition to providing the new definition of convolution and discussing its properties, we provide the spectral analysis of the convolution output. This convolutional framework can be useful in filtering applications for signals defined on the 2-sphere.**

***Index Terms*—Commutative convolution, convolution, spherical harmonics, 2-sphere (unit sphere).**

I. Introduction

In many applications in physical sciences and engineering, the domain of signals under investigation is defined on the 2-sphere, $\mathbb{S}^2$. These applications include geophysics [1], cosmology [2], electromagnetic inverse problems [3], medical imaging [4] and wireless communication systems [5]. It is often required that signal processing techniques developed for the Euclidean domain be extended and tailored in non-trivial ways so that they are suitable and well-defined for the spherical domain. One important signal processing tool is convolution between two signals defined on the 2-sphere, which is fundamental for filtering applications.

It turns out that an analog of the Euclidean-domain convolution on the 2-sphere does not exist yet in the literature. While there are various formulations [4], [6]–[10], they lack some desired or expected properties as we explain below.

One well-known and widely-used definition for convolution on the 2-sphere appears in [6], which has been generalized for the n-sphere and applied for estimation of probability density function in [11]. The advantage of this convolution is that it results in a simple multiplication of the spectral (spherical harmonic) coefficients of the signal and filter in the Fourier domain. However, the convolution involves full rotation of the filter by all independent Euler angles which includes an extra averaging over the first rotation about the $z-$ axis. This is presumably done to ensure that the output domain of convolution is $\mathbb{S}^2$, but it results in smoothing the filter by projecting it into the subspace of azimuthally symmetric signals. Consequently, this convolution becomes identical to a simpler isotropic convolution [9], [10] as shown in [12]. In contrast to conventional convolution in the Euclidean domain, due to excessive smoothing, convolution in [6], [9], [10] is not commutative and discards information.

Another definition of convolution for signals on the 2-sphere can be found in [4], [7] and has been referred to as directional correlation in

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[8], since it preserves the directional features of both the signal and filter. The convolution involves full rotation of the filter by all independent Euler angles and a double integration over the points on the 2-sphere. While this results in a desired directional or anisotropic convolution, the output remains a function of the three rotations applied to the filter and hence, its domain does not belong to $\mathbb{S}^2$. Moreover, the convolution is not commutative.

In this work, we propose a new definition of convolution on the 2-sphere that is analogous to the familiar Euclidean-domain convolution in many ways. We prove that not all independent Euler rotations should be involved in the definition of convolution. Instead, we introduce a controlled dependency between the two rotations of the filter about the $z-$ axis. Therefore, there are only two degrees of freedom in the convolution. We highlight that the same philosophy has been rightfully applied to convolution in the 2-dimensional Euclidean space, $\mathbb{R}^2$, where not all three proper "isometries" of the space (2 translations and one rotation) are involved in convolution. Our proposed definition has the following desired properties:

- It is formulated as a double integral over $\mathbb{S}^2$, which agrees with the fact that $\mathbb{S}^2$ is two dimensional;
- It generates an output whose domain remains in $\mathbb{S}^2$;
- It is anisotropic in nature, i.e., the directional features of both filter and signal contribute towards the output of convolution;
- It is commutative. That is, changing the roles of filter and signal does not change the outcome of convolution.

After introducing signals on the 2-sphere and briefly presenting some preliminaries in Section II, we review the existing definitions in the literature, which leads to our problem formulation in Section III. In Section IV, we establish a commutative anisotropic convolution and provide some graphical depiction of the proposed approach. Finally, in Section V, we present the spectral analysis of the proposed convolution.

II. Signals on the 2-Sphere

The 2-sphere, denoted by $\mathbb{S}^2$, is defined as $\mathbb{S}^2 \triangleq \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| = 1\}$. Unit vectors belonging to $\mathbb{S}^2$ are denoted by $\hat{\boldsymbol{x}} \equiv \hat{\boldsymbol{x}}(\theta,\phi) \triangleq (\sin\theta\cos\phi, \quad \sin\theta\sin\phi, \quad \cos\theta)'$ and by $\hat{\boldsymbol{y}} \equiv \hat{\boldsymbol{y}}(\vartheta,\varphi) \triangleq (\sin\vartheta\cos\varphi, \quad \sin\vartheta\sin\varphi, \quad \cos\vartheta)'$, where $(\cdot)'$ denotes vector transpose. $\theta, \vartheta \in [0,\pi]$ denote the co-latitude measured with respect to the positive $z-$ axis and $\phi,\varphi \in [0,2\pi)$ denote the longitude and are measured with respect to the positive $x-$ axis in the $x-y$ plane. In this paper, we deal with complex-valued square integrable functions whose domain is $\mathbb{S}^2$. The Hilbert space of such functions is denoted by $L^2(\mathbb{S}^2)$ with the inner product defined as

$$\langle f,h\rangle \triangleq \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{h(\hat{\boldsymbol{x}})}\,ds(\hat{\boldsymbol{x}}), \tag{1}$$

which induces the norm $\|f\| \triangleq \langle f,f\rangle^{\frac{1}{2}}$, and where $ds(\hat{\boldsymbol{x}}) = \sin\theta\,d\theta\,d\phi$ is the differential area element and the integration is carried out over the whole 2-sphere. A finite energy function in $L^2(\mathbb{S}^2)$ with $\|f\| < \infty$ is referred to as a "signal on the 2-sphere" or simply a "signal". In the following, filter h is also a signal, which is used as the kernel for convolution.

A. Spherical Harmonics

The spherical harmonic function $Y_\ell^m(\theta,\phi)$ for degree $\ell \geq 0$ and order $|m| \leq \ell$ is defined as [13], [14]

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)e^{im\phi}, \tag{2}$$

where P_ℓ^m denotes the associated Legendre function of degree ℓ and order m [14]. With the above definitions, spherical harmonic functions, or simply spherical harmonics, form a basis for $L^2(\mathbb{S}^2)$. By their completeness, any finite energy signal $f \in L^2(\mathbb{S}^2)$ can be expanded as

$$f(\hat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(f\Big)_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}), \tag{3}$$

where equality is understood in the sense of the convergence in the mean and

$$\Big(f\Big)_\ell^m \triangleq \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{Y_\ell^m(\hat{\boldsymbol{x}})}\,ds(\hat{\boldsymbol{x}}). \tag{4}$$

is the spherical harmonic Fourier coefficient, or spherical harmonic coefficient for short, of degree ℓ and order m. In the sequel, we may use the shorthand notation $\sum_{\ell,m}$ instead of $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}$.

Note that if $f(\hat{\boldsymbol{x}})$ is an azimuthally symmetric function (independent of ϕ), then only order $m=0$ spherical harmonic coefficients are non-zero. Define the projection operator $\mathcal{P}^0$, which projects a given function $f \in L^2(\mathbb{S}^2)$ onto the subspace formed by zero order spherical harmonics as

$$\Big(\mathcal{P}^0 f\Big)(\hat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty}\Big(f\Big)_\ell^0 Y_\ell^0. \tag{5}$$

The subspace formed by zero-order spherical harmonics is complete, which stems from the completeness of Legendre polynomials, P_ℓ^0. Furthermore, the projection operator $\mathcal{P}^0$ produces an azimuthally symmetric function $f_0 \triangleq \mathcal{P}^0 f$ and the operator action is understood as averaging $f(\hat{\boldsymbol{x}})$ over all longitude angles ϕ [14]. We also note the conjugate symmetry property of spherical harmonics, $\overline{Y_\ell^m(\theta,\phi)} = (-1)^m Y_\ell^{-m}(\theta,\phi)$.

B. Rotation on the 2-Sphere

Define the rotation operator by $\mathcal{D}(\varphi,\vartheta,\omega)$, which rotates a function on the sphere in a sequence of $\omega \in [0,2\pi)$ rotation about $z-$ axis, $\vartheta \in [0,\pi]$ rotation about $y-$ axis and then $\varphi \in [0,2\pi)$ rotation about $z-$ axis, which are called Euler angles. The inverse of $\mathcal{D}(\varphi,\vartheta,\omega)$ denoted by $\mathcal{D}(\varphi,\vartheta,\omega)^{-1}$ is $\mathcal{D}(-\omega,-\vartheta,-\varphi)$. If a function $f(\theta,\phi)$ is rotated on the sphere, then

$$\Big(\mathcal{D}(\varphi,\vartheta,\omega)f\Big)(\hat{\boldsymbol{x}}) = f(\boldsymbol{R}^{-1}\hat{\boldsymbol{x}}), \tag{6}$$

where $\boldsymbol{R}$ is the 3×3 rotation matrix corresponding to the rotation operator $\mathcal{D}(\varphi,\vartheta,\omega)$ [14]. The spherical harmonic coefficient of the rotated output of degree ℓ and order m is a linear combination of different order spherical harmonic coefficients of the *same* degree of the original function f as follows

$$\Big(\mathcal{D}(\varphi,\vartheta,\omega)f\Big)_\ell^m \triangleq \Big\langle \mathcal{D}(\varphi,\vartheta,\omega)f, Y_\ell^m\Big\rangle = \sum_{m'=-\ell}^{\ell} D_\ell^{m,m'}(\varphi,\vartheta,\omega)\Big(f\Big)_\ell^{m'}, \tag{7}$$

where $D_\ell^{m,m'}(\varphi,\vartheta,\omega)$ is the Wigner-D function given by [13]

$$D_\ell^{m,m'}(\varphi,\vartheta,\omega) = e^{-im\varphi}d_\ell^{m,m'}(\vartheta)e^{-im'\omega}, \tag{8}$$

and $d_\ell^{mm'}(\vartheta)$ is the Wigner-d function [13] given by

$$d_\ell^{m,m'}(\vartheta) = \sum_n (-1)^{n-m'+m} \times \frac{\sqrt{(\ell+m')!(\ell-m')!(\ell+m)!(\ell-m)!}}{(\ell+m'-n)!(n)!(\ell-n-m)!(n-m'+m)!} \times \left(\cos\frac{\vartheta}{2}\right)^{2\ell-2n+m'-m}\left(\sin\frac{\vartheta}{2}\right)^{2n-m'+m}, \tag{9}$$

where the summation over n is such that the factorial terms in the denominator remain non-negative. We also note the following relation between spherical harmonic function Y_ℓ^m and Wigner-D function, $D_\ell^{m,m'}$

$$\overline{Y_\ell^m(\vartheta,\varphi)} = \sqrt{\frac{2\ell+1}{4\pi}} D_\ell^{m,0}(\varphi,\vartheta,0). \tag{10}$$

III. Literature Review and Problem Formulation

The conventional convolution between two functions on 2-dimensional Euclidean space, $\mathbb{R}^2$ is

$$\Big(f \star h\Big)(\boldsymbol{x}) \triangleq \int_{\mathbb{R}^2} f(\boldsymbol{x}-\boldsymbol{y})h(\boldsymbol{y})\,d\boldsymbol{y}, \tag{11}$$

where $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$. It is easy to verify that the convolution is commutative, $f \star h = h \star f$.

On the premise that rotations on the sphere are counterparts of translations in the Euclidean domain, the following definitions of convolution on $\mathbb{S}^2$ in the literature involve all three independent rotations in the rotation group SO(3). The aim of the next two subsections is to formally, albeit briefly, introduce these definitions and point out the differences in their characterizations. We then pose a set of questions in search for a counterpart of Euclidean convolution on the 2-sphere.

A. Type 1 (Anisotropic) Convolution

The following definition has appeared in [4], [7]

$$g(\varphi,\vartheta,\omega) = h \boxdot f \triangleq \int_{\mathbb{S}^2} \Big(\mathcal{D}(\varphi,\vartheta,\omega)h\Big)(\hat{\boldsymbol{x}})\,f(\hat{\boldsymbol{x}})\,ds(\hat{\boldsymbol{x}}). \tag{12}$$

By this definition, the domain of convolution output does not belong to $\mathbb{S}^2$. Instead, as it is clear from above, g is a function of three independent Euler rotation angles φ,ϑ,ω. Since a proper rotation on 2-sphere is an isometry, we can apply the inverse of rotation operator to both parts of the integrand in (12) and leave the integral unchanged, as follows:

$$\begin{aligned} h \boxdot f &= \int_{\mathbb{S}^2} \Big(\mathcal{D}(\varphi,\vartheta,\omega)^{-1}\,\mathcal{D}(\varphi,\vartheta,\omega)h\Big)(\hat{\boldsymbol{x}}) \\ &\quad\times \Big(\mathcal{D}(\varphi,\vartheta,\omega)^{-1} f\Big)(\hat{\boldsymbol{x}})\,ds(\hat{\boldsymbol{x}}) \\ &= \int_{\mathbb{S}^2} h(\hat{\boldsymbol{x}})\,\Big(\mathcal{D}(\varphi,\vartheta,\omega)^{-1} f\Big)(\hat{\boldsymbol{x}})\,ds(\hat{\boldsymbol{x}}) \\ &= \int_{\mathbb{S}^2} h(\hat{\boldsymbol{x}})\,\Big(\mathcal{D}(-\omega,-\vartheta,-\varphi)\,f\Big)(\hat{\boldsymbol{x}})\,ds(\hat{\boldsymbol{x}}). \end{aligned} \tag{13}$$

However, since $\mathcal{D}(\varphi,\vartheta,\omega) \neq \mathcal{D}(-\omega,-\vartheta,-\varphi)$ in general, this convolution is not commutative.

B. Type 2 (Isotropic) Convolution

The following definition is adapted from [6]

$$\Big(h \circledcirc f\Big)(\hat{\boldsymbol{x}}) \triangleq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} h(\boldsymbol{R}^{-1}\hat{\boldsymbol{x}})f(\boldsymbol{R}\hat{\boldsymbol{\eta}})\,d\omega\,\sin\vartheta\,d\vartheta\,d\varphi, \tag{14}$$

where $\hat{\boldsymbol{\eta}} = (1,0,0)' \in \mathbb{S}^2$ is the north pole. Noting that in $f(\boldsymbol{R}\hat{\boldsymbol{\eta}})$, the first rotation by ω of the north pole around the $z-$ axis is ineffectual, we can rewrite the above convolution as

$$\Big(h \circledcirc f\Big)(\hat{\boldsymbol{x}}) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \Big(\mathcal{D}(\varphi,\vartheta,\omega)h\Big)(\hat{\boldsymbol{x}}) \times f(\vartheta,\varphi)\,d\omega\,\sin\vartheta\,d\vartheta d\varphi. \tag{15}$$

Compared to (12), (15) has a somewhat similar spirit with the difference that an extra averaging over the first rotation ω is performed which turns the filter h into an azimuthally symmetric kernel $h_0 \triangleq \mathcal{P}^0 h$. This will bring the output of the convolution back to $\mathbb{S}^2$. However, as shown in [12] this definition is identical to the isotropic convolution [9], [10]

$$(h_0 \circledast f)(\hat{\boldsymbol{x}}) \triangleq \int_{\mathbb{S}^2} h_0(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})\,f(\hat{\boldsymbol{y}})\,ds(\hat{\boldsymbol{y}}), \quad \hat{\boldsymbol{x}} \in \mathbb{S}^2, \tag{16}$$

and the extra averaging over ω "kills" any directional azimuthal component of the filter.

This convolution when evaluated in spherical harmonic domain is given by

$$\Big\langle h \circledcirc f, Y_\ell^m\Big\rangle = \Big\langle h_0 \circledast f, Y_\ell^m\Big\rangle = \sqrt{\frac{4\pi}{2\ell+1}}\Big(h\Big)_\ell^0\Big(f\Big)_\ell^m, \tag{17}$$

which has a desirable multiplicative property between spherical harmonic coefficients of the filter and signal. However, as expected, only the zero-order spherical harmonic coefficients of the filter are present, which makes this definition not commutative in general, and information is discarded.

C. Problem Formulation

From the discussion above, it becomes clear that existing definitions are either anisotropic, but with an output whose domain is not in $\mathbb{S}^2$, or their output domain is $\mathbb{S}^2$, but can only accommodate isotropic filters. None of the convolutions are commutative.

Motivated by the differences in the characterization of convolution on the sphere, we ask the following question: Is it possible to have a convolution on $\mathbb{S}^2$, which simultaneously satisfies the following requirements: 1) the domain of its output is $\mathbb{S}^2$, 2) involves a double integral over points on $\mathbb{S}^2$, 3) is anisotropic, and 4) is commutative?

Fortunately, the answer to this question is positive and we now introduce such a convolution.

IV. Commutative Anisotropic Convolution on the 2-Sphere

The Euclidean convolution in (11) forms an implicit prescription for constructing a suitable notion of convolution on $\mathbb{S}^2$. In particular, we are guided by the fact that in $\mathbb{R}^2$, not all three isometries are involved in the convolution. Only two translations and not the rotation are used. Hence, it is natural to think that only two degrees of freedom in the rotations on $\mathbb{S}^2$ should be used to define the convolution, because $\mathbb{S}^2$ is a 2-dimensional, albeit curved, surface. Therefore, we propose the following formulation as the initial candidate for our convolution

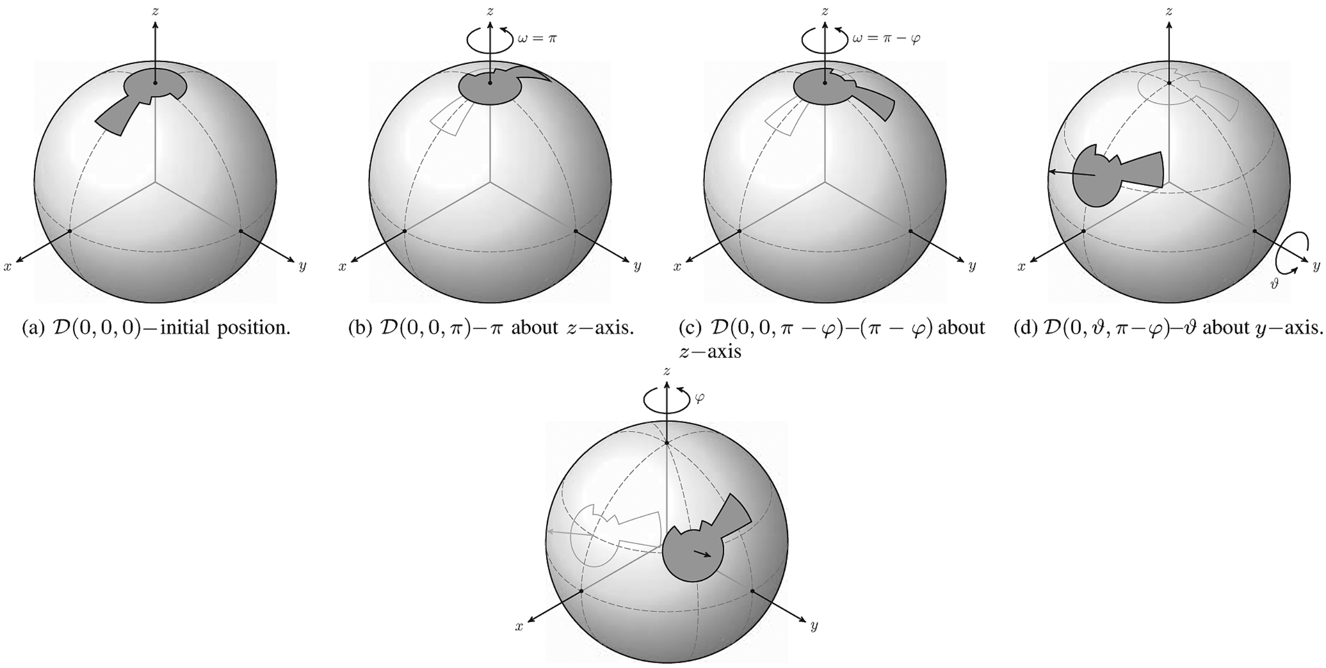
$$g_\omega(\vartheta,\varphi) \triangleq \int_{\mathbb{S}^2} \Big(\mathcal{D}(\varphi,\vartheta,\omega)h\Big)(\hat{\boldsymbol{x}})f(\hat{\boldsymbol{x}})\,ds(\hat{\boldsymbol{x}}). \tag{18}$$

This candidate appears to be somewhat similar to the anisotropic convolution in (12), but it differs in philosophy and actual content. For example, in(12), the left hand side is a function of φ,ϑ,ω or the convolution results in a function whose domain is not $\mathbb{S}^2$. Here, on the other hand, the output should be understood as a function of ϑ and φ only. The initial rotation angle ω in (18) is unspecified at this point. It might be a constant or a function of ϑ and φ. Our initial candidate satisfies the first three requirements in Section III-C. However, it is still lacking the commutative property, which will be dealt with below.

A. Commutative Anisotropic Convolution

Our aim here is to constrain the rotation operator in (18) such that the definition of convolution becomes commutative. We present the result in the following theorem.

Theorem 1: A necessary and sufficient condition for the anisotropic convolution in (18) to be commutative is $\omega = \pi - \varphi$.



(a) $\mathcal{D}(0,0,0)$—initial position. (b) $\mathcal{D}(0,0,\pi)$—π about z—axis. (c) $\mathcal{D}(0,0,\pi-\varphi)$—$(\pi-\varphi)$ about z—axis (d) $\mathcal{D}(0,\vartheta,\pi-\varphi)$—$\vartheta$ about y—axis.

(e) $\mathcal{D}(\varphi,\vartheta,\pi-\varphi)$—$\varphi$ about z—axis

Fig. 1. Action of the commutative convolution kernel. A nominal asymmetrical support region for the kernel is transformed under the action of operator $\mathcal{D}(\varphi,\vartheta,\pi-\varphi)$ according to its component rotations.

Proof: We first prove the necessary condition and show that for the convolution to be commutative, ω has to be $\pi-\varphi$. Following a similar reasoning as in (13), we can write(18) as

$$\begin{aligned} g_\omega(\vartheta,\varphi) &= \int_{\mathbb{S}^2} \Big(\mathcal{D}(\varphi,\vartheta,\omega)\, h\Big)(\hat{\boldsymbol{x}})\, f(\hat{\boldsymbol{x}})\, ds(\hat{\boldsymbol{x}}) \\ &= \int_{\mathbb{S}^2} h(\hat{\boldsymbol{x}}) \Big(\mathcal{D}(-\omega,-\vartheta,-\varphi) f\Big)(\hat{\boldsymbol{x}})\, ds(\hat{\boldsymbol{x}}). \end{aligned} \tag{19}$$

Our objective is to select ω such that

$$\mathcal{D}(\varphi,\vartheta,\omega) = \mathcal{D}(-\omega-\vartheta,-\varphi), \tag{20}$$

so that the convolution becomes commutative. The negative rotation $-\vartheta$ around $y-$ axis appears to be out of range of permissible co-latitude rotations ($[0,\pi]$), which is resolved through the following identity [14]

$$\mathcal{D}(\varphi,\vartheta,\omega) = \mathcal{D}(\pi+\varphi,-\vartheta,\pi+\omega), \tag{21}$$

where the $z-$ axis rotations have $(\mathrm{mod}\ 2\pi)$ omitted to avoid clutter. Now by equating (20) and (21), we obtain two equations $-\varphi = \pi+\omega$ and $-\omega = \pi+\varphi$, which give the value of ω

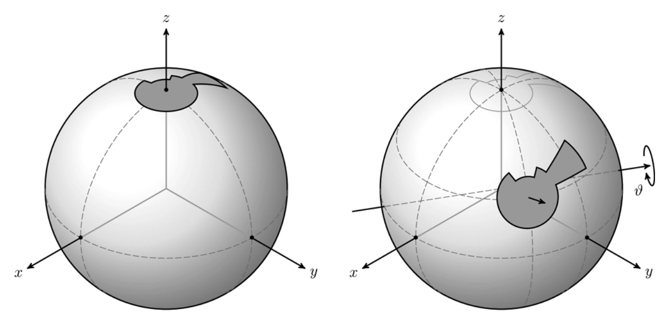
$$\omega \equiv -\pi-\varphi \equiv \pi-\varphi \pmod{2\pi}, \tag{22}$$

that makes the rotation operator $\mathcal{D}(\varphi,\vartheta,\omega)$ satisfy the "involution" property

$$\mathcal{D}(\varphi,\vartheta,\pi-\varphi) = \mathcal{D}(\varphi,\vartheta,\pi-\varphi)^{-1}.$$

Due to this involution of the rotation operator for $w = \pi-\varphi$, the anisotropic convolution in (19) becomes commutative. In fact, using a new operator $\odot$ for such a commutative convolution

$$\Big(h\odot f\Big)(\vartheta,\varphi) \triangleq g_\omega(\vartheta,\varphi)\Big|_{\omega=\pi-\varphi}, \tag{23}$$



(a) (b)

Fig. 2. Single intrinsic rotation version of $\mathcal{D}(\varphi,\vartheta,-\varphi)$. (a) Support region of some $f(\hat{\boldsymbol{x}})$. (b) Intrinsic rotation $\Big(\mathcal{D}(\varphi,\vartheta,-\varphi)f\Big)(\hat{\boldsymbol{x}})$.

we conclude that

$$\begin{aligned} \Big(h\odot f\Big)(\vartheta,\varphi) &= \int_{\mathbb{S}^2} \Big(\mathcal{D}(\varphi,\vartheta,\pi-\varphi)\, h\Big)(\hat{\boldsymbol{x}}) f(\hat{\boldsymbol{x}})\, ds(\hat{\boldsymbol{x}}) \\ &= \int_{\mathbb{S}^2} h(\hat{\boldsymbol{x}}) \Big(\mathcal{D}(\varphi,\vartheta,\pi-\varphi)^{-1} f\Big)(\hat{\boldsymbol{x}}) ds(\hat{\boldsymbol{x}}) \\ &= \int_{\mathbb{S}^2} h(\hat{\boldsymbol{x}}) \Big(\mathcal{D}(\varphi,\vartheta,\pi-\varphi)\, f\Big)(\hat{\boldsymbol{x}}) ds(\hat{\boldsymbol{x}}) \\ &= \Big(f\odot h\Big)(\vartheta,\varphi). \end{aligned}$$

The sufficient condition is proven by inserting $\omega = \pi-\varphi$ in $\mathcal{D}(\varphi,\vartheta,\omega)$ and verifying using (21) that $\mathcal{D}(\varphi,\vartheta,\pi-\varphi)^{-1} = \mathcal{D}(-\pi+\varphi,-\vartheta,-\varphi) = \mathcal{D}(\varphi,\vartheta,\pi-\varphi)$, which results in a commutative convolution. ■

Remark 1: The proposed convolution can be interpreted as a mapping of anisotropic convolution defined on SO(3) in (12) to $\mathbb{S}^2$ with the constraint that ω varies with the longitude φ given by (22). Since, ω

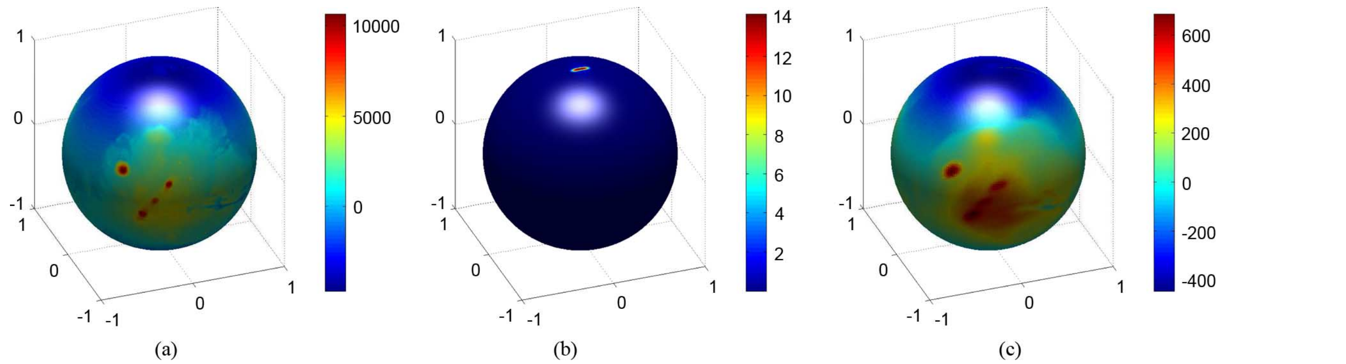


Fig. 3. Illustration of the commutative convolution. (a) Mars signal is convolved with (b) asymmetric spatially concentrated bandlimited filter kernel to obtain (c) smoothed (low-pass filtered) Mars signal. (a) Mars signal. (b) Asymmetric filter function. (c) Filtered signal.

must be chosen as a function of φ, it cannot be freely controlled at each spatial position.

In summary, we began with (18) with ω unspecified and have shown that it must be chosen to be a function of φ (and not ϑ) according to (22) for the overall rotation operation to be an involution, which yields the desired commutativity. We now provide a geometric interpretation of this definition and later we present an example to illustrate the proposed convolution.

1) Graphical Depiction: In Fig. 1 we present a sequence of images to depict the commutative convolution in action. Fig. 1(a) depicts a simplified filter signal indicated by an asymmetric region on the 2-sphere. The filter, of course, in general has support on the whole 2-sphere. The first portion of the ω rotation (by π) is shown in Fig. 1(b) and may be associated with flipping or "reversing" the filter (similar to Euclidean convolution). Fig. 1(c) is the $\omega = \pi - \varphi$ rotation; Fig. 1(d) shows the ϑ rotation and, finally, Fig. 1(e) shows the filter after the φ rotation. Fig. 2 shows an intrinsic rotation along axis $\hat{\boldsymbol{w}} = (\cos(\pi + \varphi),\ \sin(\pi + \varphi),\ 0)'$ by a single rotation ϑ, which affects the same rotation of Fig. 1(b) to Fig. 1(e).

2) Illustration: As an illustration of the proposed convolution, we consider the filtering of a signal using a filter kernel, which is spatially concentrated in an asymmetric region around north pole. We consider the Mars topographical map as a signal on the sphere which is obtained by using a spherical harmonic model for the topography of Mars[1] up to a maximum spherical harmonic degree 150 and is shown in Fig. 3(a). We use a bandlimited filter kernel with the maximum spherical harmonic degree 80 and spatial concentration in a strip region around north pole, bounded by co-latitude $\theta = \frac{\pi}{128}$ and the planes $|y| = \arcsin\frac{9\pi}{256}$ [15]. The filter kernel is obtained as a solution of Slepian concentration problem on the sphere and is shown in Fig. 3(b), which is convolved with the signal (Mars topographic map) to obtain the filtered signal shown in Fig. 3(c). This filtering is equivalent to directional smoothing of the Mars signal using asymmetric smoothing function where the direction of the smoothing function at each spatial position (ϑ, φ) is $\pi - \varphi$.

B. Alternative Characterization of Anisotropic Convolution

The expression in (18) represents the convolution in spatial domain based on the spatial-domain representations of filter and signal. Before concluding this section, we give an alternative form expressed in terms of the spherical harmonic coefficients of $h(\hat{\boldsymbol{x}})$, denoted by $\left(h\right)_s^t$, and those of $f(\hat{\boldsymbol{x}})$, denoted by $\left(f\right)_p^q$. Using the spherical harmonic coefficient expansion of f and h in (3) and the effect of rotation operator on spherical harmonic coefficients in (7), we can write (18) as

$$\begin{aligned} g_\omega(\vartheta, \varphi) &= \int_{\mathbb{S}^2} \sum_{s,t} Y_s^t(\hat{\boldsymbol{x}}) \sum_{t'=-s}^{s} D_s^{t,t'}(\varphi, \vartheta, \omega) \left(h\right)_s^{t'} \\ &\quad \times \sum_{p,q} \left(f\right)_p^q Y_p^q(\hat{\boldsymbol{x}})\, ds(\hat{\boldsymbol{x}}) \\ &= \sum_{s,t} \sum_{p,q} (-1)^q \left(f\right)_p^{-q} \delta_{s,p}\delta_{t,q} \sum_{t'=-s}^{s} D_s^{t,t'}(\varphi, \vartheta, \omega) \left(h\right)_s^{t'}, \end{aligned} \tag{24}$$

where $\delta_{s,p}$ is the Kronecker delta function and is equal to one only when $s = p$ and zero otherwise. We have also employed the conjugate symmetry property of spherical harmonics and the orthonormal property of spherical harmonics. Simplifying the expression in (24) yields the sought result

$$g_\omega(\vartheta, \varphi) = \sum_{s,t} (-1)^t \left(f\right)_s^{-t} \sum_{t'=-s}^{s} D_s^{t,t'}(\varphi, \vartheta, \omega) \left(h\right)_s^{t'}. \tag{25}$$

We note that the expression in (25) is valid for any ω (a constant or a function of ϑ and φ). The corresponding expression for commutative anisotropic convolution can be obtained by setting $\omega = \pi - \varphi$.

Let L_f and L_h denote the maximum degree of signals f and h, respectively. Then the proposed convolution can be evaluated in $O(L_g^3 \log L_g)$, where $L_g = \min(L_f, L_h)$, by employing the factoring of a single rotation into two rotations [7], followed by the use of FFT. The proposed convolution can be computed more efficiently than type 1 (anisotropic) convolution, which has the complexity of $O(L^4)$[7].

V. Spectral Analysis of Commutative Anisotropic Convolution

We now analyze the result of commutative anisotropic convolution in spherical harmonic domain. That is, we want to evaluate $\left(g\right)_\ell^m = \langle g_{\pi-\varphi}, Y_\ell^m\rangle$. We start with the expression of g_ω in (25), which specifies the anisotropic convolution output function in terms of Wigner-D functions and the spherical harmonic coefficients $\left(f\right)_s^{-t}$ and $\left(h\right)_s^t$. The spherical harmonic coefficient of the convolution output is then

$$\langle g_{\pi-\varphi}, Y_\ell^m\rangle = \sum_{s,t} (-1)^t \left(f\right)_s^{-t} \sum_{t'=-s}^{s} \left(h\right)_s^{t'} \langle D_s^{t,t'}, Y_\ell^m\rangle_{S^2}, \tag{26}$$

where the notation $\langle D_s^{t,t'}, Y_\ell^m\rangle_{S^2}$ is used to emphasize that the inner product is taken over 2 rotation angles characterizing $\mathbb{S}^2$ and not over all three rotation angles that specify SO(3). Now, using the Wigner-D

[1]http://www.ipgp.fr/~wieczor/SH/

function expression in (8) and its relation with spherical harmonics in (10), we can write $\langle D_s^{t,t'}, Y_\ell^m \rangle$ as

$$\begin{aligned}\langle D_s^{t,t'}, Y_\ell^m \rangle &= \int_0^\pi \int_0^{2\pi} D_s^{t,t'}(\varphi,\vartheta,\pi-\varphi)\overline{Y_\ell^m(\vartheta,\varphi)}\, d\varphi\, \sin\vartheta\, d\vartheta \\ &= \sqrt{\frac{2\ell+1}{4\pi}} \int_0^\pi d_s^{t,t'}(\vartheta) d_\ell^{m,0}(\vartheta) \\ &\quad \times \int_0^{2\pi} e^{-i(t-t'+m)\varphi} e^{-it'\pi} d\varphi \sin\vartheta d\vartheta, \qquad (27)\end{aligned}$$

where upon orthogonality of exponentials over $[0, 2\pi]$, the inner integral is non-zero only when $t - t' + m = 0$ or only when $t' = t + m$ and the expression in (26) can be written as

$$\begin{aligned}\langle g_{\pi-\varphi}, Y_\ell^m \rangle = (-1)^m 2\pi \sqrt{\frac{2\ell+1}{4\pi}} \sum_{s=0}^{\infty} \sum_{t=-K_1}^{K_2} \Big(f\Big)_s^{-t} \\ \times \Big(h\Big)_s^{t+m} \int_0^\pi d_s^{t,t+m}(\vartheta)\, d_\ell^{m,0}(\vartheta)\, \sin\vartheta\, d\vartheta, \qquad (28)\end{aligned}$$

where $K_1 = \max(-s, -s-m)$ and $K_2 = \min(s, s-m)$, which ensure $|m+t| \le s$. The integral in (28) that involves the inner product of Wigner-d functions can be solved using two approaches. The first approach is based on using the following expansion of product of Wigner-d functions [16]

$$\begin{aligned}\int_0^\pi d_s^{t,t+m}(\vartheta) d_\ell^{m,0}(\vartheta) \sin\vartheta d\vartheta \\ = \sum_{j=|s-\ell|}^{s+\ell} C_1(j,s,t,\ell,m) \int_0^\pi d_j^{t+m,t+m}(\vartheta) \sin\vartheta d\vartheta, \qquad (29)\end{aligned}$$

where $C_1(j,s,t,\ell,m)$ is given using Wigner-$3j$ symbols as

$$\begin{aligned}C_1(j,s,t,\ell,m) = 2j+1 \begin{pmatrix} s & \ell & j \\ t+m & 0 & -(t+m)\end{pmatrix} \\ \times \begin{pmatrix} s & \ell & j \\ t & m & -(t+m)\end{pmatrix}. \qquad (30)\end{aligned}$$

Using the expression of Wigner-d function in (9), the integral in (29) simplifies to

$$\begin{aligned}\int d_j^{t+m,t+m}(\vartheta)\, \sin\vartheta\, d\vartheta = 2 \sum_{n=0}^{\substack{\min(j+t+m,\\ j-t-m)}} (-1)^n \\ \times \frac{(j+t+m-n+1)!(j-t-m-n+1)!(j-n)!}{n!(j+1)!}. \qquad (31)\end{aligned}$$

Another approach to solve the integral in (28) is to directly use Wigner-d function in (9) to obtain

$$\begin{aligned}\int_0^\pi d_s^{t,t+m}(\vartheta)\, d_\ell^{m,0}(\vartheta)\, \sin\vartheta\, d\vartheta = \sum_n \sum_{n'} (-1)^{(n+n')} \\ \times C_2(s,t,m,n)\, C_3(\ell,m,n')\, C_4(s,\ell,n,n'), \qquad (32)\end{aligned}$$

with

$$\begin{aligned}C_2(s,t,m,n) &\triangleq \frac{\sqrt{(s+t+m)!(s-t-m)!(s+t)!(s-t)!}}{(s+t+m-n)!(n)!(s-n-t)!(n-m)!}, \\ C_3(\ell,m,n') &\triangleq \frac{\sqrt{(\ell)!(\ell)!(\ell+m)!(\ell-m)!}}{(\ell-n')!(n')!(\ell-n'-m)!(n'+m)!}, \\ C_4(s,\ell,n,n') &\triangleq 2\frac{(s+\ell-n-n')!(n+n')!}{(s+\ell+1)!},\end{aligned}$$

where the range of summations over n and n' are given by $\max(0,m) \le n \le \min(s-t, s-t+m)$ and $\max(0,-m) \le n' \le \min(\ell, \ell-m)$ with the further condition that $n+n' \le s+\ell$. The expression in (32) allows us to explicitly evaluate the integral involved in (28) without using Wigner-$3j$ symbols.

A. Special Case—One Function is Azimuthally Symmetric

Let $h(\hat{\boldsymbol{x}})$ be an azimuthally symmetric function, which implies $\Big(h\Big)_s^t = \langle h, Y_s^t \rangle = 0$ for $t \neq 0$. For this special case, we can write (25) with $\omega = \pi - \varphi$ as

$$\begin{aligned}\Big(h \odot f\Big)(\vartheta,\varphi) = g_{\pi-\varphi}(\vartheta,\varphi) = \sum_{s,t} (-1)^t d_s^{t,0}(\vartheta) \\ \times e^{-it\varphi} \Big(f\Big)_s^{-t} \Big(h\Big)_s^0, \qquad (33)\end{aligned}$$

which can be expressed using relation (10) as

$$\Big(h \odot f\Big)(\vartheta,\varphi) = \sum_{s,t} (-1)^t \sqrt{\frac{4\pi}{2s+1}} \overline{Y_s^t(\vartheta,\varphi)} \Big(f\Big)_s^{-t} \Big(h\Big)_s^0. \qquad (34)$$

Using conjugate symmetry and orthonormal property of spherical harmonics, we obtain

$$\langle h \odot f, Y_\ell^m \rangle = \sqrt{\frac{4\pi}{2\ell+1}} \Big(h\Big)_\ell^0 \Big(f\Big)_\ell^m, \qquad (35)$$

which is also equal to $\langle f \odot h, Y_\ell^m \rangle$. By commutativity, a simplified form of multiplication in spherical harmonic domain results if either the signal (nominally f) or filter (nominally h) is azimuthally symmetric.

In comparison, the convolution in [6] is not commutative and we observe

$$\begin{aligned}\langle h \circledcirc f, Y_\ell^m \rangle &= \sqrt{\frac{4\pi}{2\ell+1}} \Big(h\Big)_\ell^0 \Big(f\Big)_\ell^m, \\ \langle f \circledcirc h, Y_\ell^m \rangle &= \sqrt{\frac{4\pi}{2\ell+1}} \Big(f\Big)_\ell^0 \Big(h\Big)_\ell^0 \delta_{m,0},\end{aligned}$$

where $h(\hat{\boldsymbol{x}})$ is an azimuthally symmetric function.

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New Recursive Fast Radix-2 Algorithm for the Modulated Complex Lapped Transform

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***Abstract*—A new recursive fast radix-2 algorithm for an efficient computation of the modulated complex lapped transform (MCLT) is presented. Based on a new proposed alternative recursive sparse matrix factorization for the MDCT (modified discrete cosine transform) matrix and a relation between the MDCT and the MDST (modified discrete sine transform), firstly a new recursive fast radix-2 MDST algorithm is derived. The corresponding fast MDCT and MDST computational structures are regular and complementary to each other. Consequently, this fact enables us by their composition to construct a fast MCLT computational structure representing the fast recursive radix-2 MCLT algorithm. The fast MCLT computational structure is regular and all its stages may be realized in parallel. Combining the proposed fast radix-2 MCLT algorithm with an existing generalized fast mixed-radix MDCT algorithm defined for the composite lengths $N = 2 \times q^m, m \geq 2$, where q is an odd positive integer, we can compute the MCLT for the composite lengths $N = 2^n \times q^m, n, m \geq 2$, thus supporting a wider range of transform sizes compared to existing fast MCLT algorithms.**

***Index Terms*—Modified discrete cosine transform, modified discrete sine transform, modulated complex lapped transform, modulated lapped transform, recursive fast algorithm, recursive sparse matrix factorization.**

I. Introduction

The modulated complex lapped transform (MCLT) [1], [2] is a complex filter bank mapping overlapped blocks of real-valued signal into blocks of complex-valued transform coefficients. Its real part is the modulated lapped transform (MLT) [3], or equivalently, the modified discrete cosine transform (MDCT) [4], and its imaginary part is the corresponding modified discrete sine transform (MDST) [14]. Both the MDCT and MDST are perfect reconstruction cosine- and sine-modulated filter banks based on the concept of time domain aliasing cancellation (TDAC). The MCLT is a particular kind of $2\times$ oversampled generalized DFT filter bank [1]. Indeed, the MDCT and the MDST are, respectively, the real and imaginary components of the modified odd-time odd-frequency discrete Fourier transform ($O^2\mathrm{DFT}$) [11]. The MCLT supports the magnitude and phase representation of the signal in frequency domain which are useful measures in many perceptual audio coders for spectral analysis. Thanks to the coefficient stability and aliasing free property, the MCLT has been successfully applied to a number of audio coding problems such as audio noise reduction, acoustic echo cancellation [1], spectral adjustment and channel coupling [6], audio watermarking technology [7], [26], audio packet loss concealment [8], and acoustic data transmission systems [9], [10].

A number of fast algorithms has been developed up to now for the efficient computation of the MCLT [1], [13]–[20]. The MDCT/MDST transform kernels are cosine/sine functions, and obviously, the efficient MCLT computation may be realized indirectly via other discrete sinusoidal unitary transform of the same or reduced size such as the DFT [13]–[15], generalized discrete Hartley transform of type II (GDHT-II) [20], discrete cosine/sine transform of type IV (DCT-IV/DST-IV) [1], discrete cosine/sine transform of type II (DCT-II/DST-II) [19], or by DCT-IV and DCT-II combination [17], [18]. Note that based on a relation between the MDST and the MDCT [14], [22], the MDST computation can be realized by any existing fast MDCT computational structure with simple pre-processing of data sequences. But, a fast MDCT computational structure should be applied sequentially to obtain the MCLT. The conventional approach is to decompose the argument of MCLT transform kernel and map it into the complex DFT of the same size with complex pre- and post-multiplications [13]. However, this approach involves redundant arithmetic operations. To reduce the arithmetic complexity, the simultaneous MDCT/MDST computation is mapped into the real-valued generalized DFT of type IV (GDFT-IV) which is realized by a fast computational structure consisting of pre-butterfly stage, $\frac{N}{2}$-point DCT-IV, $\frac{N}{2}$-point DST-IV and post-butterfly stage [14]. It is well known in theory of fast MDCT/MDST algorithms that by a composition of simple permutations applied to the input data sequence, an N-point MDCT/MDST can always be converted to $\frac{N}{2}$-point DCT-IV/DST-IV (whereby DST-IV may always be converted back to DCT-IV) [1]. Note that each $\frac{N}{2}$-point DCT-IV may be alternatively mapped into the $\frac{N}{4}$-point complex DFT with identical pre- and post-rotation stages [12]. In [17], [18] ([18] is almost equivalent to [17]) by using trigonometric identities the MCLT transform kernel is decomposed into two butterfly stages and two identical $\frac{N}{2}$-point DCTs-IV, and each $\frac{N}{2}$-point DCT-IV is subsequently converted to DCT-II of the same size at the cost of additional multiplications and recursive additions. It is widely accepted that the DCT-IV-based fast MDCT/MDST algorithms are the most efficient both in terms of the arithmetic complexity and structural simplicity [22]. Taking advantage of the explicit form of the sine windowing function in the analysis/synthesis MCLT filter banks and combining it with the MCLT transform kernel, several fast MCLT algorithms have been proposed: the real-valued DFT-based [15], DCT-II-based [19] and GDHT-II-based ones [20]. They are computationally very efficient, however, their using in real audio coding applications is rather limited. Quite different fast MDCT/MDST algorithm is based on recursive sinusoidal formulas [16]. Although it is not so efficient in terms of the arithmetic complexity, it is represented by simple and regular regressive filter structures providing the variable-length MCLT computation particularly suitable for a parallel VLSI implementation.

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