

Multiscale Optimal Filtering on the Sphere

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Abstract—We present a framework for the optimal filtering of spherical signals contaminated by realizations of zero-mean anisotropic noise processes. Filtering is performed in the wavelet domain given by the scale-discretized wavelet transform on the sphere. The proposed filter is optimal in the sense that it minimizes the mean square error between the filtered wavelet representation and wavelet representation of the noise-free signal. We also present a simplified formulation of the filter for the case when azimuthally symmetric wavelet functions are used. We demonstrate the use of the proposed optimal filter for denoising of an Earth topography map in the presence of uncorrelated, zero-mean, White Gaussian noise. The proposed filter is found to be superior to the hard thresholding method, particularly in the high noise regime.

Index Terms—Wavelet transform, 2-sphere, spherical signals, anisotropic random process, spherical harmonics.

I. INTRODUCTION

Signals exhibiting angular dependence are naturally defined on the surface of the sphere and are called spherical signals. Such signals arise in different fields of science and engineering, such as astronomy [1], [2], cosmology [3]–[7], geodesy [8], computer graphics [9], [10], medical imaging [11], acoustics [12], [13] and wireless communication [14], [15]. Signal acquisition in all of these application areas is influenced by unwanted, yet unavoidable, noise which places signal estimation from its noise-contaminated samples at the heart of signal processing techniques. There is an abundance of literature on signal filtering and estimation methods [6], [16]–[23]. Some of these methods process signals using spatial (temporal) or spectral representation while others use joint spatial (temporal)-spectral representation.

Signal filtering and estimation has also been carried out using the joint scale-space representation in the Euclidean domain [24], [25]. Such a representation is enabled by the wavelet transform [26]–[28], which has proved invaluable in the multiscale analysis of one dimensional and multidimensional Euclidean domain signals. The framework for wavelet transform on the sphere has been extensively investigated in the literature as well [29]–[38]. Since dilation on the sphere, unlike in the Euclidean setting, can be defined in more than one way, there are different formulations and algorithms for multiscale analysis of spherical signals [32], [33], [35], [36], [38].

Signal estimation using wavelet transform is based on the observation that noise has a distributed representation in the wavelet domain whereas signals of interest are typically

sparsely represented, which can be exploited using different thresholding methods [39], [40]. Multiscale signal estimation in the Euclidean domain has also been carried out using signal and noise statistics through Wiener filtering [24]. In this work, we adopt the philosophy of Wiener filtering to propose a multiscale filter for signal estimation by using the scaled-discretized wavelet transform on the sphere [36]–[38]. The designed filter is optimal in the sense that the filtered wavelet representation is the minimum mean square error estimate of the scale-discretized wavelet transform of the noise-free signal. Before presenting the design of the optimal filter, we review the fundamentals of signal analysis on the sphere and $\mathbb{SO}(3)$ rotation group in Section II. We briefly review the scale-discretized wavelet transform and present the optimal filtering framework in Section III. In Section IV, we illustrate the use of the proposed multiscale optimal filtering for denoising of the bandlimited Earth topography map in the presence of uncorrelated, zero-mean, White Gaussian noise. We also test the robustness of the proposed filter at different noise levels and show that the proposed filter performs better than the hard thresholding method for signal denoising, before making concluding remarks in Section V.

II. MATHEMATICAL PRELIMINARIES

A. Signals on 2-Sphere

Spherical domain, also referred to as 2-sphere (or just sphere) is defined as $\mathbb{S}^2 \triangleq \{\hat{\mathbf{x}} \in \mathbb{R}^3 : |\hat{\mathbf{x}}| = 1\}$, where $|\cdot|$ represents the Euclidean norm. A point $\hat{\mathbf{x}} \in \mathbb{S}^2$ can be represented as $\hat{\mathbf{x}}(\theta, \phi) \triangleq (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T$, where $(\cdot)^T$ denotes vector transpose and the angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ are referred to as colatitude and longitude and are measured from the positive z -axis and positive x -axis (in the $x - y$ plane) respectively. Complex-valued and square-integrable functions defined on the sphere form a Hilbert space $L^2(\mathbb{S}^2)$, which is equipped with the following inner product for $f, g \in L^2(\mathbb{S}^2)$

$$\langle f, g \rangle_{\mathbb{S}^2} \triangleq \int_{\mathbb{S}^2} f(\hat{\mathbf{x}}) \overline{g(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}), \quad \int_{\mathbb{S}^2} \triangleq \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi}, \quad (1)$$

where $\overline{(\cdot)}$ denotes complex conjugate and $ds(\hat{\mathbf{x}}) = \sin \theta d\theta d\phi$. Norm of the function f is induced as $\|f\|_{\mathbb{S}^2} = \sqrt{\langle f, f \rangle_{\mathbb{S}^2}}$ and its energy¹ is given by $\langle f, f \rangle_{\mathbb{S}^2}$. For $L^2(\mathbb{S}^2)$, spherical harmonics, denoted by $Y_{\ell}^m(\hat{\mathbf{x}})$ for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$, form a complete set of orthonormal basis functions [41], which can be used to represent any signal $f \in L^2(\mathbb{S}^2)$ as

$$f(\hat{\mathbf{x}}) = \sum_{\ell, m} (f)_{\ell}^m Y_{\ell}^m(\hat{\mathbf{x}}), \quad \sum_{\ell, m} \equiv \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}, \quad (2)$$

¹We refer to finite energy functions as signals.

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where $(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle_{\mathbb{S}^2}$ is the spherical harmonic (spectral) coefficient of degree ℓ and order m and constitutes the spectral domain representation of the signal f . Signal f is considered bandlimited to degree L if $(f)_\ell^m = 0$ for $\ell \geq L$.

B. Signals on $\mathbb{SO}(3)$ Rotation Group

Rotations on the sphere are specified by three Euler angles namely, $\omega \in [0, 2\pi)$, $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$, using the right handed zyz convention, in which a point on the sphere is sequentially rotated by ω , ϑ and φ around z , y and z axes respectively. Group of all proper rotations², specified by the three Euler angles $(\varphi, \vartheta, \omega)$, is called the Special Orthogonal group, denoted by $\mathbb{SO}(3)$. Each point in the $\mathbb{SO}(3)$ rotation group is represented by a 3-tuple of Euler angles as $\rho = (\varphi, \vartheta, \omega)$. We define rotation of signals on the sphere through an operator $\mathcal{D}(\rho)$. Spectral representation of a signal $f \in L^2(\mathbb{S}^2)$ under the action of $\mathcal{D}(\rho)$ is given by

$$(\mathcal{D}f)_\ell^m \triangleq \langle \mathcal{D}f, Y_\ell^m \rangle_{\mathbb{S}^2} = \sum_{m'=-\ell}^{\ell} D_{m,m'}^\ell(\rho) (f)_{\ell}^{m'}, \quad (3)$$

where $D_{m,m'}^\ell(\rho)$ is called the Wigner- D function of integer degree $\ell \geq 0$ and integer orders $|m|, |m'| \leq \ell$ [41].

For square-integrable and complex-valued functions, of the form $g(\rho)$, defined on the $\mathbb{SO}(3)$ rotation group, we define the inner product between a pair of such functions, g, ν , as

$$\langle g, \nu \rangle_{\mathbb{SO}(3)} \triangleq \int_{\mathbb{SO}(3)} g(\rho) \overline{\nu(\rho)} d\rho, \quad \int_{\mathbb{SO}(3)} \triangleq \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{\omega=0}^{2\pi}, \quad (4)$$

where $d\rho = d\varphi \sin \vartheta d\vartheta d\omega$. Equipped with the inner product in (4), the set of signals defined on the $\mathbb{SO}(3)$ rotation group form a Hilbert space, denoted by $L^2(\mathbb{SO}(3))$, which has Wigner- D functions as basis functions that satisfy the following orthogonality relationship

$$\langle D_{m,m'}^\ell, D_{q,q'}^p \rangle_{\mathbb{SO}(3)} = c_\ell \delta_{\ell,p} \delta_{m,q} \delta_{m',q'}, \quad c_\ell \triangleq \frac{8\pi^2}{2\ell+1}. \quad (5)$$

Hence, we can express any signal $g \in L^2(\mathbb{SO}(3))$ as

$$g(\rho) = \sum_{\ell,m,m'} (g)_{\ell,m,m'}^\ell \overline{D_{m,m'}^\ell(\rho)}, \quad \sum_{\ell,m,m'} \equiv \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell}, \quad (6)$$

where $(g)_{\ell,m,m'}^\ell = (1/c_\ell) \langle g, \overline{D_{m,m'}^\ell} \rangle_{\mathbb{SO}(3)}$ forms the spectral domain representation of the signal g .

III. OPTIMAL FILTERING USING SCALE-DISCRETISED WAVELET TRANSFORM

We consider a realization of an anisotropic random process as the source signal $s(\hat{\mathbf{x}})$, which is contaminated by a realization of a zero-mean anisotropic random process, called the noise signal $z(\hat{\mathbf{x}})$. The source and noise signals are assumed to be uncorrelated, i.e., $\mathbb{E}\{(s)_\ell^m \overline{(z)_\ell^q}\} = 0$, where $\mathbb{E}\{\cdot\}$ represents the expectation operator. We further assume that the spectral covariances of the source and noise processes, denoted by $\mathbf{C}^s$, $\mathbf{C}^z$ respectively and defined as $C_{\ell m, pq}^s =$

$\mathbb{E}\{(s)_\ell^m \overline{(s)_\ell^q}\}$, $C_{\ell m, pq}^z = \mathbb{E}\{(z)_\ell^m \overline{(z)_\ell^q}\}$, are known. Given the noise-contaminated observation $f(\hat{\mathbf{x}}) = s(\hat{\mathbf{x}}) + z(\hat{\mathbf{x}})$, the problem under consideration is to obtain an estimate of the source signal, $\tilde{s}(\hat{\mathbf{x}})$, by filtering the observation f at different scales using the scale-discretized wavelet transform. Before we design an optimal filter for signal estimation, we present a multiscale representation of signals in the wavelet domain.

A. Scale-Discretised Wavelet Transform on 2-Sphere

In this section, we briefly review the directional scale-discretized wavelet transform given in [36]–[38]. For a signal $f \in L^2(\mathbb{S}^2)$, the directional scale-discretized wavelet transform is defined as

$$w_f^{\Psi^{(j)}}(\rho) = \langle f, \mathcal{D}_\rho \Psi^{(j)} \rangle_{\mathbb{S}^2} = \sum_{\ell,m,m'} (f)_\ell^m \overline{(\Psi^{(j)})_{\ell}^{m'}} \overline{D_{m,m'}^\ell(\rho)}, \quad (7)$$

where $w_f^{\Psi^{(j)}} \in L^2(\mathbb{SO}(3))$ is the wavelet coefficient of the signal f , j is the discrete wavelet scale, $(\Psi^{(j)})_{\ell}^{m'}$ is the spectral coefficient of the wavelet function $\Psi^{(j)} \in L^2(\mathbb{S}^2)$ and we have used (3) along with the orthonormality of spherical harmonics to evaluate the second equality. Wavelet functions capture the high frequency (rapid variations along the space) content of the signal f . Low frequency content of the signal is represented by the scaling coefficient defined as

$$w_f^\Phi(\hat{\mathbf{x}}) = \langle f, \mathcal{D}_{\hat{\mathbf{x}}} \Phi \rangle_{\mathbb{S}^2} = \sum_{\ell,m} \sqrt{\frac{4\pi}{2\ell+1}} (f)_\ell^m \overline{(\Phi)_\ell^0} Y_\ell^m(\hat{\mathbf{x}}), \quad (8)$$

for the azimuthally symmetric scaling function $\Phi \in L^2(\mathbb{S}^2)$, where $\mathcal{D}_{\hat{\mathbf{x}}} \triangleq \mathcal{D}(\varphi, \vartheta, 0)$ and we have used the following relation between Wigner- D function and spherical harmonics

$$D_{\ell,0}^\ell(\varphi, \vartheta, 0) = \sqrt{\frac{4\pi}{2\ell+1}} \overline{Y_\ell^m(\varphi, \vartheta)}. \quad (9)$$

Signal f can be reconstructed perfectly from its wavelet and scaling coefficients using the following expression

$$f(\hat{\mathbf{x}}) = \int_{\mathbb{S}^2} w_f^\Phi(\hat{\mathbf{y}}) (\mathcal{D}_{\hat{\mathbf{y}}} \Phi)(\hat{\mathbf{x}}) ds(\hat{\mathbf{y}}) + \int_{\mathbb{SO}(3)} \sum_{j=j_1}^{j_2} w_f^{\Psi^{(j)}}(\rho) (\mathcal{D}_\rho \Psi^{(j)})(\hat{\mathbf{x}}) d\rho, \quad (10)$$

provided the following admissibility condition holds

$$c_\ell \left(\frac{1}{2\pi} |(\Phi)_\ell^0|^2 + \sum_{j=j_1}^{j_2} \sum_{m'=-\ell}^{\ell} |(\Psi^{(j)})_{\ell}^{m'}|^2 \right) = 1, \quad \forall \ell, \quad (11)$$

where j_1 and j_2 denote the minimum and maximum wavelet scale³ respectively and this condition can be derived from (10) using (7) and (8). Wavelet and scaling functions, which satisfy the admissibility condition, have the following spectral representation

$$(\Psi^{(j)})_{\ell}^m = \frac{1}{\sqrt{c_\ell}} \Gamma_\Psi^{(j)}(\ell, \gamma) \zeta_\ell^m, \quad (\Phi)_\ell^m = \sqrt{\frac{2\pi}{c_\ell}} \Gamma_\Phi(\ell, \gamma), \quad (12)$$

such that $\sum_{m=-\ell}^{\ell} |\zeta_\ell^m|^2 = 1$ for all values of ℓ for which ζ_ℓ^m is non-zero for at least one value of m , and $|\Gamma_\Phi|^2 + \sum_{j=j_1}^{j_2} |\Gamma_\Psi^{(j)}|^2 = 1$. Here c_ℓ is given in (5), γ is the harmonic space dilation

²An improper rotation is a reflection about either an axis or the center of the coordinate system.

³Wavelet scale is inversely proportional to the degree ℓ .

parameter, $\Gamma_{\Psi}^{(j)}$, Γ_{Φ} are the harmonic tiling functions, which control the angular localization, and ζ_{ℓ}^m encodes the directional features of the wavelet functions.⁴

1) *Azimuthally Symmetric Wavelet Functions*: Azimuthally symmetric wavelet functions are defined by the following spectral representation [41]

$$\left(\Psi^{(j)}\right)_{\ell}^0 = \left(\Psi^{(j)}\right)_{\ell}^m \delta_{m,0}, \quad (13)$$

due to which the wavelet coefficients are given by [40]

$$w_f^{\Psi^{(j)}}(\hat{\mathbf{x}}) = \left\langle f, \mathcal{D}_{\hat{\mathbf{x}}} \Psi^{(j)} \right\rangle_{\mathbb{S}^2} = \sum_{\ell,m}^{\infty} \sqrt{\frac{4\pi}{2\ell+1}} (f)_{\ell}^m \overline{(\Psi^{(j)})_{\ell}^0} Y_{\ell}^m(\hat{\mathbf{x}}).$$

Signal f can be reconstructed from the following inverse transform

$$f(\hat{\mathbf{x}}) = \int_{\mathbb{S}^2} \left(w_f^{\Phi}(\hat{\mathbf{y}}) (\mathcal{D}_{\hat{\mathbf{y}}} \Phi)(\hat{\mathbf{x}}) + \sum_{j=j_1}^{j_2} w_f^{\Psi^{(j)}}(\hat{\mathbf{y}}) (\mathcal{D}_{\hat{\mathbf{y}}} \Psi^{(j)})(\hat{\mathbf{x}}) \right) ds(\hat{\mathbf{y}}),$$

if the following admissibility condition holds [40]

$$\frac{c_{\ell}}{2\pi} \left(|(\Phi)_{\ell}^0|^2 + \sum_{j=j_1}^{j_2} |(\Psi^{(j)})_{\ell}^0|^2 \right) = 1, \quad \forall \ell. \quad (14)$$

B. Optimal Filtering in the Wavelet Domain

We define a joint $\mathbb{SO}(3)$ -scale domain filter function as

$$\Xi(\rho; j) = \sum_{\ell, m, m'}^{L_{\Xi_j}-1} (\Xi(\cdot; j))_{m, m'}^{\ell} \overline{D_{m, m'}^{\ell}(\rho)}, \quad (15)$$

for wavelet scales $j = j_1, \dots, j_2$. Action of this filter on wavelet coefficients of the noise-contaminated observation f is given by $\mathbb{SO}(3)$ convolution, which is defined as [42], [43]

$$w_s^{\Psi^{(j)}}(\rho) = \left(\Xi(\cdot; j) \otimes w_f^{\Psi^{(j)}} \right)(\rho) = \int_{\mathbb{SO}(3)} \Xi(\rho \rho_1^{-1}; j) w_f^{\Psi^{(j)}}(\rho_1) d\rho_1,$$

where $\tilde{s}(\hat{\mathbf{x}})$ is the source signal estimate and $\otimes$ denotes convolution on $\mathbb{SO}(3)$ rotation group. Using the addition formula for Wigner- D functions [44], $\mathbb{SO}(3)$ convolution can be written in a computationally more amenable form as [43]

$$w_s^{\Psi^{(j)}}(\rho) = \sum_{\ell, m, m'}^{L-1} c_{\ell} \sum_{k=-\ell}^{\ell} (f)_{\ell}^k (\Xi(\cdot; j))_{m, k}^{\ell} \overline{(\Psi^{(j)})_{\ell}^{m'} D_{m, m'}^{\ell}(\rho)}, \quad (16)$$

where c_{ℓ} is given in (5), L is the bandlimit of signal f and we have assumed, without loss of generality, that the bandlimit of the filter function at each scale is equal to the signal bandlimit, i.e., $L_{\Xi} = L$. We design an optimal filter that minimizes the joint $\mathbb{SO}(3)$ -scale domain mean square error given by

$$\mathcal{E}_{ms} = \mathbb{E} \left\{ \sum_{j=j_1}^{j_2} \|w_s^{\Psi^{(j)}}(\rho) - w_s^{\Psi^{(j)}}(\rho)\|_{\mathbb{SO}(3)}^2 \right\}. \quad (17)$$

We present the result in the following theorem.

Theorem 1. *Let the source signal $s(\hat{\mathbf{x}})$ be a realization of an anisotropic random process which is contaminated by a realization of a zero-mean anisotropic noise process, $z(\hat{\mathbf{x}})$, to obtain the observation $f(\hat{\mathbf{x}}) = s(\hat{\mathbf{x}}) + z(\hat{\mathbf{x}})$. The source and*

noise signals are uncorrelated with known spectral covariance matrices, defined as $C_{\ell m, pq}^s = \mathbb{E}\{(s)_{\ell}^m \overline{(s)_{\ell}^p}\}$, $C_{\ell m, pq}^z = \mathbb{E}\{(z)_{\ell}^m \overline{(z)_{\ell}^p}\}$. Defining c_{ℓ} as in (5), spectral coefficients of the joint $\mathbb{SO}(3)$ -scale domain filter in (15), which minimize the joint $\mathbb{SO}(3)$ -scale domain mean square error in (17), can be obtained by inverting the following linear system

$$\mathbf{A}^T(\ell) \Xi(j, \ell, m) = \mathbf{b}(\ell, m), \quad (18)$$

for $|m| \leq \ell$, $0 \leq \ell \leq L-1$, $j_1 \leq j \leq j_2$, where elements of the column vector $\Xi(j, \ell, m)$ are given $\Xi_k = (\Xi(\cdot; j))_{m, k}^{\ell}$, $|k| \leq \ell$, and elements of the matrix $\mathbf{A}$ and column vector $\mathbf{b}$ are given by

$$A_{k, k'} = c_{\ell} \sum_{k=-\ell}^{\ell} (C_{\ell k, \ell k'}^s + C_{\ell k, \ell k'}^z), \quad |k|, |k'| \leq \ell, \quad (19)$$

$$b_{k'} = C_{\ell m, \ell k'}^s, \quad |k'| \leq \ell. \quad (20)$$

Proof. Using (7) and (16), along with the orthogonality of Wigner- D functions on the $\mathbb{SO}(3)$ rotation group, the mean square error in (17) can be written as

$$\mathcal{E}_{ms} = \sum_{j=j_1}^{j_2} \sum_{\ell, m, m'}^{L-1} c_{\ell} \times \mathbb{E} \left\{ \left(c_{\ell} \sum_{k=-\ell}^{\ell} (f)_{\ell}^k \overline{(\Psi^{(j)})_{\ell}^{m'}} (\Xi(\cdot; j))_{m, k}^{\ell} - (s)_{\ell}^m \overline{(\Psi^{(j)})_{\ell}^{m'}} \right) \times \overline{\left(c_{\ell} \sum_{k'=-\ell}^{\ell} (f)_{\ell}^{k'} \overline{(\Psi^{(j)})_{\ell}^{m'}} (\Xi(\cdot; j))_{m, k'}^{\ell} - (s)_{\ell}^m \overline{(\Psi^{(j)})_{\ell}^{m'}} \right)} \right\}.$$

Differentiating $\mathcal{E}_{ms}$ with respect to $(\Xi(\cdot; j))_{m, k'}^{\ell}$ and setting the result equal to zero, we obtain a linear system which, using (19) and (20), can be cast in the matrix form in (18). $\square$

Having found the spectral representation of the joint $\mathbb{SO}(3)$ -scale domain filter, signal estimate, $\tilde{s}(\hat{\mathbf{x}})$, is obtained from the wavelet coefficients in (16) using (10).

Remark 1. *For azimuthally symmetric wavelet functions, the joint spatial-scale domain filter function is defined as*

$$\Xi(\hat{\mathbf{x}}; j) = \sum_{\ell, m}^{L_{\Xi_j}-1} (\Xi(\cdot; j))_{\ell}^m Y_{\ell}^m(\hat{\mathbf{x}}), \quad (21)$$

and its action on the noise-contaminated observation f is given by the following $\mathbb{S}^2$ convolution [45]

$$w_s^{\Psi^{(j)}}(\hat{\mathbf{x}}) = \sum_{\ell, m}^{L-1} \left(\sqrt{\frac{4\pi}{2\ell+1}} (f)_{\ell}^m \overline{(\Psi^{(j)})_{\ell}^0} \right) (\Xi(\cdot; j))_{\ell}^m Y_{\ell}^m(\hat{\mathbf{x}}), \quad (22)$$

which necessitates the minimization of the following joint spatial-scale domain mean square error for finding the spectral coefficients of the filter function in (21)

$$\mathcal{E}_{ms} = \mathbb{E} \left\{ \sum_{j=j_1}^{j_2} \|w_s^{\Psi^{(j)}}(\hat{\mathbf{x}}) - w_s^{\Psi^{(j)}}(\hat{\mathbf{x}})\|_{\mathbb{S}^2}^2 \right\}. \quad (23)$$

From the relation between Wigner- D functions and spherical harmonics in (9), it can be seen that (22) can be obtained from (16) by setting $m' = 0$ and $(\Xi(\cdot; j))_{m, k}^{\ell} =$

⁴See [36], [38] for details on the construction of harmonic tiling functions and directionality component of the wavelet and scaling functions.

$(1/c_\ell)(\Xi(\cdot; j))_\ell^m \delta_{m,k}$. Hence, by setting $k = k' = m$ in (19) and (20), spectral coefficients of the filter in (21) can be directly obtained from (18) as

$$(\Xi(\cdot; j))_\ell^m = \frac{C_{\ell m, \ell m}^s}{(C_{\ell m, \ell m}^s + C_{\ell m, \ell m}^z)}. \quad (24)$$

IV. ANALYSIS

Performance of the optimal filter using scale-discretized wavelet transform is gauged by the signal-to-noise ratio, which for any signal $d \in L^2(\mathbb{S}^2)$ is defined as

$$\text{SNR}^d = 20 \log \frac{\|s(\hat{\mathbf{x}})\|_{\mathbb{S}^2}^2}{\|d(\hat{\mathbf{x}}) - s(\hat{\mathbf{x}})\|_{\mathbb{S}^2}^2}. \quad (25)$$

From this definition, input and output SNRs are given by SNR^f and $\text{SNR}^{\tilde{s}}$ respectively. We demonstrate the utility of the optimal filtering framework by using the Earth topography map⁵, bandlimited to degree $L = 64$, as the source signal $s(\hat{\mathbf{x}})$, and contaminating it with realizations of uncorrelated, zero-mean, White Gaussian noise process at different values of SNR^f . Spectral covariance matrix of the source signal is computed as $C_{\ell m, p q}^s = (s)_\ell^m (s)_p^q$. Spectral covariance matrix of the Gaussian noise process is generated as $\mathbf{C}^z = \sigma^2 \mathbf{I}$, where $\mathbf{I}$ is the identity matrix and σ^2 is the noise energy given by

$$\sigma^2 = 10^{-\text{SNR}^f/10} \sum_{\ell, m} |(s)_\ell^m|^2. \quad (26)$$

We use azimuthally symmetric wavelet and scaling functions for filtering and estimation of the source signal using the scale-discretized wavelet transform. At $L = 64$, the maximum wavelet scale $j_2 = 6$ [38], and we choose the minimum wavelet scale $j_1 = 0$. Dilation parameter for the harmonic tiling functions is set to 2 [38]. Fig. 1 shows an illustration of the optimal filtering framework in which the Earth topography map is contaminated with an uncorrelated, zero-mean, White Gaussian noise at $\text{SNR}^f = -0.057$ dBs. The output SNR is measured to be 9.68 dBs, resulting in a significant SNR gain of 9.7 dBs.

We compare the performance of the proposed filtering framework with the hard thresholding method for signal denoising [39], [40], by setting the threshold equal to $3\sigma_j$, where $\sigma_j^2 \triangleq \mathbb{E} \left\{ \left| w_z^{(j)}(\hat{\mathbf{x}}) \right|^2 \right\}$ is the noise variance in the wavelet domain at scale j and is given by [40]

$$\sigma_j^2 = \sigma^2 \sum_{\ell=0}^{L-1} \left| (\Psi^{(j)})_\ell^0 \right|^2, \quad \sigma^2 = \frac{\text{trace}(\mathbf{C}^z)}{L^2}. \quad (27)$$

We contaminate the Earth topography map, bandlimited to degree $L = 64$, with 10 realization of uncorrelated, zero-mean, White Gaussian noise process at different values of SNR^f , and compute $\text{SNR}^{\tilde{s}}$. Fig. 2 shows the output SNR versus input SNR, averaged over 10 realizations, in which the proposed optimal filtering method can be seen to perform better than the hard thresholding method, particularly in the low SNR regime, where a mean output SNR difference of as large as ~ 3.8 dBs can be observed at mean $\text{SNR}^f = -15$ dBs.

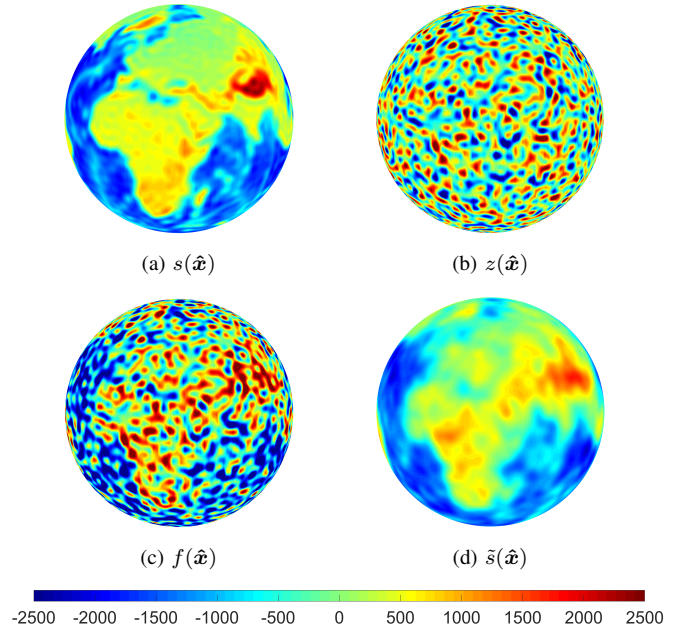


Fig. 1: Optimal filtering of the bandlimited Earth topography map using the scale-discretized wavelet transform. Earth topography map $s(\hat{\mathbf{x}})$ is contaminated with uncorrelated, zero-mean, White Gaussian noise $z(\hat{\mathbf{x}})$ at $\text{SNR}^f = -0.057$ dBs. Output SNR obtained from the source signal estimate $\tilde{s}(\hat{\mathbf{x}})$ is 9.68 dBs, resulting in an SNR gain of 9.7 dBs.

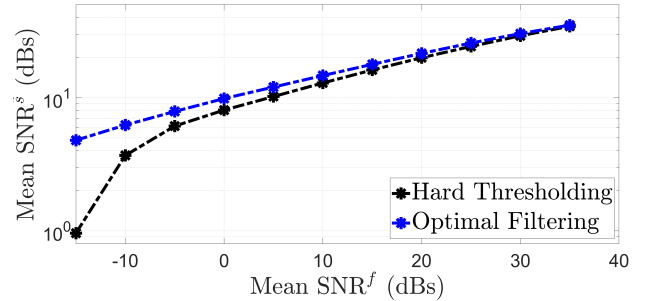


Fig. 2: Output SNR plotted against input SNR, averaged over 10 realizations of uncorrelated, zero-mean, White Gaussian noise process, for the proposed optimal filtering framework and the hard thresholding method.

V. CONCLUSION

We have formulated an optimal filter in the wavelet domain, using the scale-discretized wavelet transform, for estimation of signals which have been contaminated by realizations of a zero-mean anisotropic noise process on the sphere. The designed filter is optimal in the sense that the filtered representation in the wavelet domain is the minimum mean square error estimate of the wavelet representation of the noise-free signal. We have illustrated the utility of the proposed filtering framework on the bandlimited Earth topography map in the presence of uncorrelated, zero-mean, White Gaussian noise. We have also compared the performance of the proposed optimal filter with the hard thresholding method for signal denoising, and have shown that the optimal filter performs better, particularly at high noise levels.

⁵<http://geoweb.princeton.edu/people/simons/software.html>

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